Converting a DFA to a REG EXP
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1 The Basic Algorithm

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA. We can assume $Q = \{1, 2, \ldots, n\}$. We show how to construct a reg expression $\alpha$ that generates the same set the DFA $M$ recognizes.

Let $R(i, j, k)$ be a regular expression for the set of strings $x$ such that if you run $M$ started at state $i$, only using states $\{1, \ldots, k\}$ (or a subset of them), you end up in state $j$.

We first show how to find $R(i, j, 0)$. Then, assuming one has $R(i, j, k - 1)$ for ALL $i, j$, we derive $R(i, j, k)$ for ALL $i, j$.

$R(i, j, 0)$: Note that the only way to NOT use ANY states as intermediaries is to either transition directly from $i$ to $j$. Hence the following seems reasonable:

$$R(i, j, 0) = \{ \sigma \in \Sigma \mid \delta(i, \sigma) = j \}.$$  

This IS correct if $i \neq j$. However, if $i = j$ then the empty string also takes you from state $i$ to state $i$ without using any intermediary states. So

$$R(i, i, 0) = \{e\} \cup \{\sigma \in \Sigma \mid \delta(i, \sigma) = j\}.$$  

(NOTE: To understand this next equation you really need to be in class.)

$$R(i, j, k) = R(i, j, k - 1) \cup R(i, k, k - 1)R(k, k, k - 1)^*R(k, j, k - 1)$$

Hence, by induction on $k$, all of the $R(i, j, k)$ are regular expressions.

Assume that the start state is 1. The regular expression we seek is

$$\bigcup_{f \in F} R(1, f, n)$$