Homework 6 DUE March 31 at 11:00 AM (No Dead Cat Extension)
THIS HOMEWORK IS TWO PAGES LONG!!!!!!!!!!!!!!!

1. (0 points, but if you actually miss the midterm without telling Dr. Gasarch ahead of time, you will lose 100 points on this homework) When will the midterm be (give date and time)? When will the final be (give date and time)? By when do you have to tell Dr. Gasarch that you cannot make the midterm?

2. (40 points, 10 points each part) For each of the following say if it is

   - REGULAR (in this case give a DFA or NFA or REGEX for it)
   - CONTEXT FREE BUT NOT REGULAR (in this case give a CFG for it and prove it's not regular)
   - NOT CONTEXT FREE (in this case just say NOT a CFL, no proof required).

   Here are the languages:

   (a) $\{a^{2n}b^{3n} : n \in \mathsf{N}\}$.
   (b) $\{a^{n^2+2n} : n \in \mathsf{N}\}$.
   (c) $\{a^{n+m} : n, m \in \mathsf{N} \wedge n \geq 3m\}$
   (d) $\{a^n b^n : n \in \mathsf{N} \wedge n \leq 100\}$.

   SOLUTION TO PROBLEM 2

   (a) $\{a^{2n}b^{3n} : n \in \mathsf{N}\}$.
   CONTEXT FREE BUT NOT REGULAR.
   CONTEXT FREE:
   $S \to aaSbbb$
   $S \to e$.
   NOT REGULAR: Use the Extended Pumping Lemma with the cases of the y is all a's. Details omitted.

(b) $\{a^{n^2+2n} : n \in \mathsf{N}\}$. NOT CONTEXT FREE: A proof was not required. Even so, we give one based on the pumping theorem for Context-free grammars (which we did not present in class.) Assume it is context free . Let n be large. Then $a^{n^2+2n} = uvxyz$ where $u = a^{m_1}$, $v = a^{m_2}$, $x = a^{m_3}$, $y = a^{m_4}$, $z = a^{m_5}$. Let $m_1 + m_3 + m_5 = M$. Let $m_2 + m_4 = N$. Note that, for all i, $a^{m_1+im_2+m_3+im_4+m_5} \in L$. Hence $a^{M+iN} \in L$. Therefore

$M + N = n^2 + 2n$

$M + 2N \geq (n+1)^2 + 2(n+1)$.

$M + 3N \geq (n+2)^2 + 2(n+2)$.

$\vdots$

$(\forall i)[M + iN \geq (n+i-1)^2 + 2(n+i-1)]$

After some algebra you find that M is larger than something of the form $Ai + B$. Hence M is larger than any given number. This contradicts M being a constant.

(c) $\{a^{n+m} : n, m \in \mathsf{N} \wedge n \geq 3m\}$ REGULAR: This is just a^*.

(d) $\{a^n b^n : n \in \mathsf{N} \wedge n \leq 100\}$. REGULAR. This is a finite set, hence regular.

3. (30 points, 10 points per part) Give a CFG in Chomsky Normal Form for the languages. Try to make the the number of productions as small as possible.

(a) $L = \{a^{16}\}$

(b) $L = \{a^{17}\}$

(c) $L = \{a^{28}\}$

SOLUTION TO PROBLEM 3

(a) $L = \{a^{16}\}$

$S \to S_1 S_1$

$S_1 \to S_2 S_2$

$S_2 \to S_3 S_3$

$S_3 \to S_4 S_4$

$S_4 \to a$.

NOTE: 5 productions.

Note that

S_4 produces a.

S_3 produces S_4S_4 which produces a^2.

S_2 produces S_3S_3 which produces a^4

S_1 produces S_2S_2 which produces a^8

S produces S_1S_1 which produces a^{16}

(b) $L = \{a^{17}\}$

We need to have S produce a^{16} and THEN a.

$S \to TU$.

$T \to T_1T_1$

$T_1 \to T_2T_2$

$T_2 \to T_3T_3$

$T_3 \to T_4T_4$

$T_4 \to a$.

$U \to a$.

NOTE: 7 productions.

Note that T_3 produces a^{16} and T_4 produces a, so S produces a^{17}

(c) $L = \{a^{28}\}$

We need to have S produce a^{16} and THEN a^{11}.

$S \to TU$.

$T \to T_1T_1$

$T_1 \to T_2T_2$

$T_2 \to T_3T_3$

$T_3 \to T_4T_4$

$T_4 \to a$.

T will produce a^{16}.

Need U to produce $a^{11} = a^8a^3$.

$U \to VW$.

$V \to V_1V_1$

$V_1 \to V_2V_2$

$V_2 \to V_3V_3$

$V_3 \to a$.

V will produce a^8.

Need W to produce a^3.

$W \to XY$

$X \to XX$

$X \to a$.

$Y \to a$.

W produces a^3.

15 productions.

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4. (30 points) For any $F \subseteq \mathsf{N}$, let $BILL(F)$ be the following set of strings.

$$BILL(F) = \{a^n b^n : n \in F\}$$

(a) (8 points) TRUE or FALSE. If F is finite, then $BILL(F)$ is regular. If TRUE then show how to, given a finite F, construct a DFA for $BILL(F)$. **Give a reasonable upper bound on the number of states in the DFA as a function of** F e.g. the number of states is bounded by the largest Fibonacci prime in F (this is not the answer). If FALSE, then give a finite set F for which $BILL(F)$ is not regular and prove that it is not regular.

(b) (7 points) TRUE or FALSE. If F is finite, then $BILL(F)$ is context free. If TRUE then show how to, given a finite F, construct a CFG for $BILL(F)$. **Give a reasonble upper bound on the number of production rules in the CFG as a function of** F e.g. the number of production rules is bounded by the largest Ramanujan prime in F (this is not the answer). (The CFG NEED NOT be in Chomsky Normal Form.) If FALSE, then give a finite set F for which $BILL(F)$ is not context free and prove that it is not context free.

(c) (8 points) TRUE or FALSE. If F is infinite, then $BILL(F)$ is regular. If TRUE then show how to, given a infinite F, construct a DFA for $BILL(F)$. **Give a reasonable upper bound on the number of states in the DFA as a function of** F e.g. the number of states is the smallest Ramanujan prime in F (this is not the answer). If FALSE, then give a infinite set F for which $BILL(F)$ is not regular and prove that it is not regular.

(d) (7 points) TRUE or FALSE. If F is infinite, then $BILL(F)$ is context free. If TRUE then show how to, given a infinite F, how to construct a CFG for $BILL(F)$. **Give a reasonble upper bound on the number of production rules in the CFG as a function of** F e.g. the number of states is the smallest Fermat prime in F (this is not the answer). (The CFG NEED NOT be in Chomsky Normal Form.) If FALSE, then give a infinite set F for which $BILL(F)$ is not context free. NO need to prove it.

SOLUTION TO PROBLEM 4

(a) TRUE. Let $F = \{m_1, \ldots, m_L\}$. For each $1 \le i \le L$, construct a DFA M_i that accepts $\{a^{m_i}b^{m_i}\}$. Note that M_i has $m_i + 1$ states. Construct an NFA that has L e-transitions from the start state, the i^{th} one leading to the i^{th} DFA. Solutios with $O(L^2)$ states and with $O(L)$ states were shown in class.

(b) TRUE. Let $F = \{m_1, \ldots, m_L\}$. The following CFG generates $BILL(F)$

$S \to a^{m_1}b^{m_1}$

$S \to a^{m_2}b^{m_2}$

$\vdots$

$S \to a^{m_L}b^{m_L}$.

This CFG has L productions.

(c) FALSE. Let $F = \mathsf{N}$, then $L = \{a^n b^n : n \in \mathsf{N}\}$. We omit the proof that it is not regular.

(d) FALSE. Let F be the set of all squares.