Homework 8 Morally DUE April 28 at 11:00 AM

1. (35 points) Let

$$IS_\alpha = \{G : G \text{ has an Independent Set of size } \geq \alpha n\}$$

where n is the number of vertices in G.

(a) (15 points) Show that $IS_{1/3}$ is NP-complete. (Hint: look at the proof that IS is NP-complete.)

(b) (20 points) Show that

$$IS_{1/2} \leq IS_{7/8}$$

You can assume that the graph G you are originally given has n vertices where n is divisible by 8.

(You must give the reduction; you can't just say they are both NP-complete, though they are.)

SOLUTION

(a) Show that $IS_{1/3}$ is NP-complete. (Hint: look at the proof that IS is NP-complete.)

SOLUTION: In the proof that IS is NP-complete we proved

$$3SAT \leq IS$$

By taking a formula $C_1 \wedge \cdots \wedge C_k$, where each C_i has three literals, and forming a graph that had:

For each $1 \leq i \leq k$, three literals labelled with C_i. SO there were $3k$ vertices.

And we are looking for a k-clique, which is 1/3 times the number of variables.

(b) Show that

$$IS_{1/2} \leq IS_{7/8}$$

You can assume that the graph G you are originally given has n vertices where n is divisible by 8.

(You must give the reduction; you can't just say they are both NP-complete, though they are.)

SOLUTION

We need to show how to, given a graph G on n vertices create a graph G' on n' vertices such that

G has an IS of size $n/2$ iff G' has an IS of size $7n'/8$

We DERIVE the key parameter in the reduction and then do it with that parameter.

G' is G with x vertices (we determine x later) added that are all isolated vertices.

If G has an IS of size $n/2$ then G' has an IS of size $(n/2) + x$.

G' has $n + x$ vertices and has an IS of size $n/2 + x$.

We need

$$\frac{\frac{n}{2} + x}{n + x} = \frac{7}{8}$$

$$4n + 8x = 7n + 7x$$

$$x = 3n$$

SO: G' is G with $3n$ more vertices added all of which are isolated.

OKAY, let's now prove this works.

If G has an IS of size $n/2$ then G' has an IS of size $n/2 + 3n = \frac{7n}{2}$. G' has $n + 3n = 4n$ vertices. Note that $\frac{7n/2}{4n} = \frac{7}{8}$.

We also need the CONVERSE: If G' has an IS of size $\frac{7}{8}(n') = \frac{7}{8}(n + 3n) = \frac{7}{8}(4n) = \frac{7n}{2}$ then G has an IS of size $\frac{7n}{2} - 3n = \frac{n}{2}$ which is half the number of vertices in G. (Recall that we added $3n$ isolated vertices to make G', so the other $\frac{n}{2}$ vertices were in G to begin with.)

END OF SOLUTION

2. (30 points) In class we did the proof that $3SAT \le IS$.

   Let ϕ be

   $$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_2 \vee x_3)$$

   Apply the reduction to obtain a graph G and a number k such that

   ϕ is satisfable IFF G has an ind set of size k.

   Give the graph BOTH as a drawing and FORMALLY in terms of listing its vertices and edges.

   SOLUTION

   We omit the picture.

   We describe the graph and then give the formal description of it.

   The graph will have, for each clause, a set of vertices labeled with the literals in that clause. Each vertex will have an edge connecting to each other vertex in its clause. However, we need to make sure the vertices are distinct. Hence the vertices are

   $(1, x_1), (1, \neg x_2), (1, x_3)$

   $(2, \neg x_1), (2, x_2), (2, x_4)$

   $(3, x_1), (3, \neg x_3), (3, \neg x_4)$

   $(4, \neg x_1), (4, x_2), (4, x_3)$

   We put an edge between two vertices from different sets that conflict. We list them out in batches depending on which literal we are looking at.

   $((1, x_1), (2, \neg x_1))$

   $((1, x_1), (4, \neg x_1))$

   $((3, x_1), (2, \neg x_1))$

   $((3, x_1), (4, \neg x_1))$

   $((2, x_2), (1, \neg x_2))$

   $((4, x_2), (1, \neg x_2))$

$((1, x_3), (3, \neg x_3))$

$((4, x_3), (3, \neg x_3))$

$((2, x_4), (3, \neg x_4))$

ϕ is satisfiable IFF this graph has an independent set of size 4.

END OF SOLUTION

3. (35 points) A Sam-TM is one that allows the instruction

$$\delta(q, a) = (p, b, L)$$

which means that, if the machine is in state q and is looking at a, then the state changes to p, The a is overwritten with a b, AND the head then moves left.

Write the part of the formula that models this transition in the proof of the Cook-Levin Theorem.

SOLUTION Omitted