Closure of Regular Langs Under Union, Intersection, Complementation, and Projection
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1 Introduction

We give the constructions that show sketch the proof that all if $L_1$ and $L_2$ are regular and
$L_1 \cap L_2$, $L_1 \cup L_2$, $\overline{L}$, and $proj(L)$ (which we will define) are regular.

Def 1.1 A DFA is a tuple $(Q, \Sigma, \delta, s, F)$ where $\delta : Q \times \Sigma \rightarrow Q$.

We define running a DFA $M$ on a string $x$ in the obvious way. If the DFA ends in a state in $F$ then $x$ is accepted. Otherwise its rejected.

2 Closure Under Intersection

Theorem 2.1 If $L_1$ and $L_2$ are regular then $L_1 \cap L_2$ is regular.

Proof: Let $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ be the DFA for $L_1$. Let $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ be the DFA for $L_2$.

We define a DFA for $L_1 \cap L_2$. Let $M = (Q_1 \times Q_2, \Sigma, \delta, (s_1, s_2), F_1 \times F_2)$ where $\delta$ is defined by, for $(q_1, q_2) \in Q_1 \times Q_2$ and $\sigma \in \Sigma$,

$$\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)).$$

The intuition is that the DFA $M$ runs $M_1$ and $M_2$ at the same time. If both end up in $F_1 \times F_2$ then both $M_1$ and $M_2$ accepted.

3 Closure Under Union

Theorem 3.1 If $L_1$ and $L_2$ are regular then $L_1 \cup L_2$ is regular.

Proof: Let $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ be the DFA for $L_1$. Let $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ be the DFA for $L_2$.

We define a DFA for $L_1 \cup L_2$. Let $M = (Q_1 \times Q_2, \Sigma, \delta, (s_1, s_2), F_1 \times Q_2 \cup Q_1 \times F_2)$ where $\delta$ is defined by, for $(q_1, q_2) \in Q_1 \times Q_2$ and $\sigma \in \Sigma$,

$$\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)).$$

The intuition is that the DFA $M$ runs $M_1$ and $M_2$ at the same time. If $M_1$ ends up in $F_1$ then we accept (independent of what $M_2$ does), and if $M_2$ ends up in $F_2$ then we accept (independent of what $M_1$ does).
4 Closure Under Complementation

Theorem 4.1 If \( L \) is regular then \( \overline{L} \) is regular.

Proof:
Let \( M = (Q, \Sigma, \delta, s, F) \) be the DFA for \( L \).
We define a DFA for \( \overline{L} \). Let \( M' = (Q, \Sigma, \delta, s, Q - F) \) (recall that \( Q - F = \{ q \mid q \in Q \land q \notin F \} \)).
The intuition is that the DFA \( M' \) runs \( M \) but does the opposite when it comes to accepting.

5 Closure Under Complementation

To Compliment a DFA you say

\[ DFA, \ I \ admire \ your \ states! \]

6 NDFA’s and DFA’s

Recall the definition of an NDFA:

Def 6.1 An NDFA is a tuple \( (Q, \Sigma, \Delta, s, F) \) where \( \Delta : Q \times (\Sigma \cup \epsilon) \rightarrow 2^Q \). (Recall that \( 2^Q \) is the powerset of \( Q \).

We DO NOT define running an NDFA \( M \) on a string \( x \). Instead we say that an NDFA accepts \( x \) if SOME way of running the machine ends up in a state in \( F \).

Theorem 6.2 If \( L \) is accepted by an NDFA then there exists a DFA such that accepts \( L \).

Proof: Let \( M = (Q, \Sigma, \Delta, s, F) \) be the NDFA for \( L \).
We define a DFA for \( L \). Let \( M' = (2^Q, \Sigma, \delta, s, \mathcal{F}) \) where for \( A \in 2^Q \) and \( \sigma \in \Sigma \),

\[ \delta(A, \sigma) = \bigcup_{q \in A} \Delta(e^a q e^b, \sigma) \]
(The \( e^a \) and \( e^b \) are strings of the empty string.)

\[ \mathcal{F} = \{ A \mid A \cap F \neq \emptyset \} \]

The intuition is that the DFA \( M' \) runs ALL possibilities for \( M \). If SOME possibility ends up accepting, then accept.
7 Closure under Projection

Notation 7.1 Let $\Sigma = \{0, 1\}^n$. Note that each element of $\Sigma$ is itself a string of $n$ bits. If $x \in \Sigma^*$ then $\text{proj}(x)$ is what you get by taking each symbol in $x$ and chopping off the last bit. So if $x \in (\{0, 1\}^n)^*$ then $\text{proj}(x) \in (\{0, 1\}^{n-1})^*$. If $L \subseteq (\{0, 1\}^n)^*$ then

$$\text{proj}(L) = \{\text{proj}(x) \mid x \in L\}.$$  

Theorem 7.2 If $L$ is regular then $\text{proj}(L)$ is regular.

Proof: Let $M = (Q, \Sigma, \delta, s, F)$ be the DFA for $L$. We define an NDFA for $L$. Let $M' = (Q, \Sigma, \delta, s, F)$. For $q \in Q$ and $\sigma \in \{0, 1\}^{n-1}$

$$\Delta(q, \sigma) = \{\delta(q, \sigma 0), \delta(q, \sigma 1)\}.$$

8 Closure under Concatenation

Theorem 8.1 If $L_1$ and $L_2$ are regular then $L_1 L_2$ is regular.

Proof:

Let $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ be the DFA for $L_1$. Let $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ be the DFA for $L_2$. By relabelling we can assume $Q_1 \cap Q_2 = \emptyset$.

We define an NDFA for $L_1 L_2$. By Theorem 6.2 we could then obtain a DFA for $L_1 L_2$. Let $M = (Q_1 \cup Q_2, \Sigma, \delta, s_1, F_2)$ where $\delta$ is defined as follows:

- If $q_1 \in Q_1$ and $\sigma \in \Sigma$ then $\delta(q_1, \sigma) = \delta_1(q_1, \sigma)$.
- If $q_2 \in Q_2$ and $\sigma \in \Sigma$ then $\delta(q_2, \sigma) = \delta_2(q_2, \sigma)$.
- If $f_1 \in F_1$ then $\delta(f_1, e) = s_2$

The intuition is that the NDFA $M$ runs $M_1$ and then nondeterministically hops to $M_2$. But the hop must be from $f_1 \in F_1$ to $s_2 \in Q_2$ and then the $M_2$ must accept, so if $w$ is accepted there must be SOME WAY to divide it $w = xy$ where $x \in L_1$ and $y \in L_2$.  

9 Closure Under $^*$

Theorem 9.1 If $L$ is regular then $L^*$ is regular.

Proof:

Let $M = (Q, \Sigma, \delta, s, F)$ be the DFA for $L$. We define an NDFA for $L^*$. LEAVE AS AN EXERCISE.