### **Decidability and Undecidability**

Exposition by William Gasarch—U of MD

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- 3. If you run  $M_e(d)$  it might not halt.
- 4. Everything computable is computable by some TM.
- 5. A TM that halts on all inputs is called **total**.

### **Computable Sets**

**Definition** A set A is *computable* if there exists a Turing Machine M that behaves as follows:

$$M(x) = \begin{cases} Y & \text{if } x \in A \\ N & \text{if } x \notin A \end{cases} \tag{1}$$

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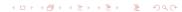
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Some examples of computable sets.

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- 2.  $\{(e,d,s): M_{e,s}(d) \downarrow \}$ .
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- Yes—ALL SETS: uncountable. DEC Sets: countable, hence there exists an uncountable number of noncomputable sets.
- 3. That last answer is true but unsatisfying. We want an actual example of an noncomputable set.

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**Recall** You all thought there was no small NFA for  $\{a^i : i \neq n\}$  and were wrong. Hence lower bounds need proof.



#### **HALT** is Undecidable

**Theorem** HALT is not computable. **Proof** Assume HALT computable via TM *M*.

$$M(e,d) = \begin{cases} Y & \text{if } M_e(d) \downarrow \\ N & \text{if } M_e(d) \uparrow \end{cases}$$
 (2)

We use M to create the following machine which is  $M_e$ .

- 1. Input *d*
- 2. Run M(d,d)
- 3. If M(d, d) = Y then RUN FOREVER.
- 4. If M(d,d) = N then HALT.

$$M_e(e) \downarrow \implies M(e,e) = Y \implies M_e(e) \uparrow$$
  
 $M_e(e) \uparrow \implies M(e,e) = N \implies M_e(e) \downarrow$ 

We now have that  $M_e(e)$  cannot  $\downarrow$  and cannot  $\uparrow$ . Contradiction.

#### Other Undecidable Problems

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Proofs by reductions. Similar to NPC. We will not do that.
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Reductions in Computability theory came first by several decades. Complexity theory borrowed ideas from Computability theory for the basic definitions.

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#### Key Difference:

- ► Semantic Question: What does M<sub>e</sub> do? is usually undecidable.
- ► Syntactic Question: What does M<sub>e</sub> look like? is usually decidable.

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- 3.  $\Sigma_1$  came first by several decades. Complexity theory borrowed ideas from Computability theory for the basic definitions.
- 4. Are ideas from Computability theory useful in complexity theory? Yes, to a limited extent. My thesis was on showing some of those limits.

## More on $\Sigma_1$

**Theorem** Let A be any set. The following are equivalent:

- (1) A is  $\Sigma_1$ .
- (2) There exists a TM such that  $A = \{x : (\exists s)[M_{e,s}(x) \downarrow]\}.$
- (3) There exists a total TM such that  $A = \{y : (\exists e, s)[M_{e,s}(x) \downarrow = y]\}.$

Because of (3)  $\Sigma_1$  is often called recursively enumerable or computably enumerable.

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 $TOT = \{x : (\forall y)(\exists s)[M_{x,s}(y) \downarrow]\} \in \Pi_2.$   
Known:  $TOT \notin \Sigma_1 \cup \Pi_1.$ 

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The proof involved coding Turing Machines into Polynomials.

Upshot This problem of, given  $p(x_1, ..., x_n) \in \mathbb{Z}[x_1, ..., x_n]$  does it have an integer solution is a natural question that is undecidable.