Reductions TO SAT
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1 Introduction

Recall

Def 1.1 $CLIQ = \{(G,k)\ G$ has a clique of size k $\}$

In class we showed SAT $\le$ CLIQ.

By the Cook-Levin theorem (which we have not proven yet), for all $A \in NP$, $A \le$ SAT. Hence CLIQ $\le$ SAT.

We will prove CLIQ $\le$ SAT directly.

2 CLIQ $\le$ SAT

Given (G,k) we want to come up with a formula ϕ such that

$$(G,k) \in \text{CLIQ IFF } \phi \in SAT$$

We restate the problem a bit.

Let K_k be the complete graph on k vertices.

Def 2.1 INJ is the set of all (G,k) such that that following is true: There is an injection f of K_k into G such that

$$\text{for all } 1 \le i \le j \le k,\ (f(i), f(j)) \in E$$

(E is the set of edges in G.)

It is easy to see that

$$(G,k) \in \text{CLIQ iff } (G,k) \in \text{INJ}$$

Theorem 2.2 INJ $\le$ SAT. *Hence* CLIQ $\le$ SAT*.*

Proof: The input is (G,k).

K_k has vertices $\{1,\ldots,k\}$ and G has vertices $\{1,\ldots,n\}$.

Our Boolean formula will have variables:

x_{ij} where $1 \le i \le k$ and $1 \le j \le n$.

The intention is that the x_{ij} code an injection by having x_{ij} is TRUE if i gets mapped to j, and FALSE otherwise.

The first few clauses will ensure that the x_{ij}'s codes an injection.

1. For all $1 \le i \le k$ the clause $(x_{i1} \vee \cdots \vee x_{in})$. Hence vertex in K_k maps to at least one vertex of G.

2. For all i, j_1, j_2, $\neg x_{ij_1} \vee \neg x_{ij_2})$. Hence vertex in K_k maps to at most one vertex of G.

The above clauses makes the x_{ij} code an injection.

We now need clauses to make sure that that K_k maps to is clique. We do this by

Let $i_1, i_2 \in \{1, \ldots, k\}$. We need that i_1 and i_2 map to two vertices that have an edge. Let E be the edges of G. Then we need

$$\bigvee_{(j_1,j_2)\in E} x_{i_1,j_1} \wedge x_{i_2,j_2}$$

SO, the final formula is:

$$\bigwedge_{i\in\{1,\ldots,k\}} x_{i1} \vee \cdots \vee x_{in}$$

$$\bigwedge$$

$$\bigwedge_{j_1,j_2\in\{1,\ldots,n\}} \bigwedge_{i\in\{1,\ldots,k\}} \neg x_{ij_1} \vee \neg x_{ij_2})$$

$$\bigwedge$$

$$\bigwedge_{i_1,i_2\in\{1,\ldots,k\}} \bigvee_{(j_1,j_2)\in E} x_{i_1,j_1} \wedge x_{i_2,j_2}$$

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