Homework 1 Morally Due Feb 8
WARNING: THIS HW IS FIVE PAGES LONG!!!!!!!!!!!!!!!!!!!!!!

1. (0 points, but if you actually miss the midterm without telling Dr. Gasarch ahead of time, you will lose 100 points on this homework)
   When will the midterm be (give date and time)? When will the final be (give date and time)? By when do you have to tell Dr. Gasarch that you cannot make the midterm?

2. (20 points) The alphabet is \{a, b\}. Write a DFA-classifier (by giving its table) which, on input \(w\), tells you
   \begin{itemize}
   \item \(\#_a(w) \mod 7\)
   \item \(\#_b(w) \mod 13\)
   \end{itemize}

   Specify \(Q\), \(s\), \(\delta\), and for each \((i, j)\) where \(0 \leq i \leq 6\) and \(0 \leq j \leq 12\), what state (or states) a string with \(\#_a(w) \equiv i \mod 7\) and \(\#_b(w) \equiv j \mod 13\) ends up. Note that \(\delta(q, \sigma)\) has to be defined for every states \(q\) and \(\sigma \in \Sigma\).

   Also note how many STATES your classifier has.

   SOLUTION
   \(Q = \{0, \ldots, 6\} \times \{0, \ldots, 12\}\).

   \(s = (0, 0)\).

   \(\delta((i, j), a) = (i + 1 \mod 7, j)\)

   \(\delta((i, j), b) = (i, j + 1 \mod 13)\)

   If \(\#_a(w) \equiv i \mod 7\) and \(\#_b(w) \equiv j \mod 13\) then the DFA ends up in state \((i, j)\).

   The DFA has 91 states.

   END OF SOLUTION

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3. (20 points) The alphabet is \{a, b\}. Write an NFA (by giving its table) which accepts the following language:

\( \{ w : \#_a(w) \equiv 3 \pmod{7} \lor \#_b(w) \equiv 4 \pmod{13} \} \)

that has FAR LESS STATES than the DFA in the last problem.

Specify \( Q, s, \Delta, F \). We demand that \( \Delta(q, \sigma) \) has to be defined for every states \( q \) and \( \sigma \in \Sigma \). (There may also be some \( \Delta(q, e) \).)

Also note how many STATES your NFA has.

**SOLUTION**

\[ Q = \{ s, d \} \cup \{ (a, 0), \ldots, (a, 6) \} \times \{ (b, 0), \ldots, (b, 12) \}. \]

\( s \) will be the start states. \( d \) will be a dump state.

Start state is \( s \)

\( \Delta(s, e) = \{(a, 0), (b, 0)\} \)

\( \Delta((a, i), a) = (a, i + 1 \pmod{7}) \)

\( \Delta((a, i), b) = d \)

\( \Delta((b, i), b) = (b, i + 1 \pmod{13}) \)

\( \Delta((b, i), a) = d \)

\( \Delta(d, a) = d \)

\( \Delta(d, b) = d \)

\( F = \{(a, 3), (b, 4)\} \)

The number of states is \( 2 + 7 + 13 = 22 \).

**END OF SOLUTION**
4. (20 points) Assume the alphabet is \( \Sigma = \{a, b\} \). Let \( L = \Sigma^*a\Sigma^3 \). Show that any DFA for \( L \) requires at least 8 states. (NOTE: This has NOTHING to do with the DFA we drew that HAS 8 states. We want to show that ANY DFA has \( \geq 8 \) states.) (HINT: Show that \( aaa, aab, \ldots, bbb \) all go to DIFFERENT states.)

**SOLUTION**

Let \( M \) be a DFA for \( L \). Let \( Q \) be its state set. Let \( \delta \) be extended to strings. We claim that the map \( \delta \) from \( \Sigma^3 \) to \( Q \) is 1-1 We look at every pair.

**Case 1** The strings differ on the first symbol.

\[
\delta(s, a\sigma_1\sigma_2) = \sigma(s, b\sigma_3\sigma_4).
\]

Hence \( \delta(s, a\sigma_1\sigma_2a) = \sigma(s, b\sigma_3\sigma_4a) \).

BUT \( \delta(s, a\sigma_1\sigma_2a) \in F \) and \( \delta(s, b\sigma_3\sigma_4a) \notin F \).

**Case 2** The strings differ on the second symbol.

\[
\delta(s, \sigma_1a\sigma_2) = \sigma(\sigma_3b\sigma_4).
\]

Hence \( \delta(s, \sigma_1a\sigma_2aa) = \sigma(\sigma_3b\sigma_4aa) \).

BUT \( \delta(s, \sigma_1a\sigma_2aa) \in F \) and \( \delta(s, \sigma_3b\sigma_4aa) \notin F \).

**Case 3** The strings differ on the third symbol.

\[
\delta(s, \sigma_1\sigma_2a) = \sigma(\sigma_3\sigma_4b).
\]

Hence \( \delta(s, \sigma_1\sigma_2aaa) = \sigma(\sigma_3\sigma_4baaa) \).

BUT \( \delta(s, \sigma_1\sigma_2aaa) \in F \) and \( \delta(s, \sigma_3\sigma_4baaa) \notin F \).

**END OF SOLUTION**

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5. (20 points) The alphabet is \{0, 1, 2, 3, 4, 5, 6\}. We view strings as numbers in base 10. We read them least-sig-digit-first. So for example, if we want to input 9318 it will go in 8 \text{ THEN } 1 \text{ THEN } 3 \text{ THEN } 9. Write the DFA-classifier (as a table) that tells us what a number is mod 7.

Specify \( Q, s, \delta \) and which state (or states) a number \( \equiv i \pmod{\text{mod})} \) ends up. Note that \( \delta(q, \sigma) \) has to be defined for \textit{every} states \( q \) and \( \sigma \in \Sigma \).

Also note how many STATES your classifier has.

\textbf{SOLUTIONS}

Omitted.

\textbf{END OF SOLUTIONS}

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6. (20 points) The alphabet is \( \{a, b\} \). Let \( x \in \Sigma^* \). Then

\( \text{SUBSEQ}(x) \) is the set of all strings formed by removing some letters
(possibly 0, possibly all of them)

Example

\[
\text{SUBSEQ}(abbaa) = \{e, a, b, aa, ab, ba, bb, aaa, aba, abb, baa, bba\}.
\]

Let \( L \) be a language.

Then

\[
\text{SUBSEQ}(L) = \bigcup_{x \in L} \text{SUBSEQ}(x).
\]

And now FINALLY for the question.

Show that if \( L \) is regular then \( \text{SUBSEQ}(L) \) is regular.

SOLUTION

Let \( M \) be the DFA for \( L \). We produce an NFA for \( \text{SUBSEQ}(L) \).

Intuitively: add to every transition and \( e \)-transition.

Formally: Let \( M = (Q, \Sigma, \delta, s, F) \).

Then \( M' \), the NFA for \( \text{SUBSEQ}(L) \) is \( (Q, \Sigma, \Delta, s, F) \) where

For all \( p \in Q \) and \( \sigma \in \Sigma \), \( \Delta(p, \sigma) = \delta(p, \sigma) \).

And also

For all \( p, q \in Q \) and for all \( \sigma \in \Sigma \), if \( \delta(p, \sigma) = q \) then

\[
\Delta(p, e) = q.
\]

END OF SOLUTION