Converting a DFA to a REG EXP: An Example
Exposition by William Gasarch

\( M = (Q, \Sigma, \delta, s, F) \) is a DFA. \( R(i, j, k) \) is a reg exp for \( \{ x \mid \delta(i, x) = j \} \).

Recall:

\[ R(i, j, 0) = \{ \sigma \in \Sigma \mid \delta(i, \sigma) = j \}. \]

\[ R(i, i, 0) = \{ e \} \cup \{ \sigma \in \Sigma \mid \delta(i, \sigma) = j \}. \]

\[ R(i, j, k) = R(i, j, k - 1) \cup R(i, k, k - 1)R(k, k, k - 1)^*R(k, j, k - 1) \]

The regular expression for the language accepted by \( M \) is \( \bigcup_{f \in F} R(1, f, n) \)

We work this out on the following small example. We will start with an NFA rather than a DFA—The NFA will leave out some transitions and hence be smaller in this case, better to make the example shorter.

\( Q = \{ 1, 2, 3 \} \)
\( s = 1 \)
\( F = \{ 3 \} \)
\( \delta(1, a) = 2 \)
\( \delta(1, b) = 3 \)
\( \delta(2, a) = 3 \)
\( \delta(2, b) \) does not exist.
\( \delta(3, a) = 3 \)
\( \delta(3, b) = 3 \)

We want to know \( R(1, 3, 3) \). Rather than compute all 3 \( \times 3 \times 4 = 36 \) \( R(i, j, k) \)’s, we see which ones we need.

ALL OF THE \( R(\cdot, \cdot, 3) \) THAT WE NEED: \( R(1, 3, 3) \). (only 1)

Since \( R(1, 3, 3) = R(1, 3, 2) \cup R(1, 3, 2)R(3, 3, 2)^*R(3, 3, 2) \)

ALL OF THE \( R(\cdot, \cdot, 2) \) THAT WE NEED: \( R(1, 3, 2), R(3, 3, 2) \). (only 2)

We need \( R(1, 3, 2) \). We use

\[ R(1, 3, 2) = R(1, 3, 1) \cup R(1, 2, 1)R(2, 2, 1)^*R(2, 3, 1) \]

Hence we need \( R(1, 3, 1), R(1, 2, 1), R(2, 2, 1), R(2, 3, 1) \).

We need \( R(3, 3, 2) \). ANOTHER SHORTCUT: Since state 3 is a self-loop it cannot ever use any other state, so \( R(3, 3, 2) = R(3, 3, 0) \). We keep this in mind for later.

ALL OF THE \( R(\cdot, \cdot, 1) \) THAT WE NEED:
\( R(1, 2, 1), R(1, 3, 1), R(2, 2, 1), R(2, 3, 1) \) (only 4).

We are not going to bother to figure out which \( R(\cdot, \cdot, 0) \) we need since its easier to just computer all nine of them. Note that we will get \( R(3, 3, 0) \) which we need.

We first look at ALL of the \( R(i, j, 0) \).

\[ R(1, 1, 0) = e \]
\[ R(1, 2, 0) = a \]
\[ R(1, 3, 0) = b \]
\[ R(2, 1, 0) = \emptyset \]
\[ R(2, 2, 0) = e \]
\[ R(2, 3, 0) = a \]
\[ R(3, 1, 0) = \emptyset \]
\[ R(3, 2, 0) = \emptyset \]
\[ R(3, 3, 0) = e \cup a \cup b \]

We now look at all of the \( R(i, j, 1) \) that we need.

\[ R(1, 2, 1) = R(1, 2, 0) \cup R(1, 1, 0)R(1, 1, 0)^*R(1, 2, 0) = a \cup ee^*a = a \]
\[ R(1, 3, 1) = R(1, 3, 0) \cup R(1, 1, 0)R(1, 1, 0)^*R(1, 3, 0) = b \cup ee^*b = b \]
\[ R(2, 2, 1) = R(2, 2, 0) \cup R(2, 1, 0)R(1, 1, 0)^*R(1, 2, 0) = e \cup \emptyset e^*a = e \cup \emptyset = e \]
\[ R(2, 3, 1) = R(2, 3, 0) \cup R(2, 1, 0)R(1, 1, 0)^*R(1, 3, 0) = a \cup \emptyset e^*b = e \cup \emptyset = a \]

We now look at all of the \( R(i, j, 2) \) that we need.

\[ R(1, 3, 2) = R(1, 3, 1) \cup R(1, 2, 1)R(2, 2, 1)^*R(2, 3, 1) = b \cup ae^*a = b \cup aa \]
\[ R(3, 3, 2) = R(3, 3, 0) = e \cup a \cup b \]

We now look at all of the \( R(i, j, 3) \), just \( R(1, 3, 3) \).

\[ R(1, 3, 3) = R(1, 3, 2) \cup R(1, 3, 2)R(3, 3, 2)^*R(3, 3, 2) = (b \cup aa) \cup (b \cup aa)(e \cup a \cup b)^*(a \cup b) \]

Reg Exp for the language is \( R(1, 3, 3) \) above.