CLIQ ≤ SAT

Exposition by William Gasarch—U of MD
CLIQ ≤ SAT. Why?

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Yaelle That’s stupid! We know CLIQ ≤ SAT by Cook-Levin.
${\text{CLIQ}} \leq {\text{SAT}}$. Why?

**Bill** Today we will prove $\text{CLIQ} \leq \text{SAT}$.

**Yaelle** That’s stupid! We know $\text{CLIQ} \leq \text{SAT}$ by Cook-Levin.

**Bill** Write a program that will, given $(G, k)$ produce $\phi$ such that

$$(G, k) \in \text{CLIQ} \text{ iff } \phi \in \text{SAT}$$
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Yaelle  Deal with Turing Machines? That’s insane!
Bill  Correct. I will show CLIQ ≤ SAT in a sane way.
Yaelle  Why? Not practical since SAT is hard. Not theoretically interesting since we already know CLIQ ≤ SAT.
Bill  Because there are awesome SAT Solvers!
Old View, New View

I want to solve CLIQ. Since $\text{SAT} \leq \text{CLIQ}$, CLIQ is probably hard. Darn!

I want to solve CLIQ. I know from Cook-Levin that $\text{CLIQ} \leq \text{SAT}$. That reduction is insane (hard and blow up). If I can find a better reduction of $\text{CLIQ} \leq \text{SAT}$ then to solve a CLIQ problem I can transform it to a SAT problem, and solve that.

Caveat

1. SAT solvers are only good on some problems.
2. Getting the reductions to not blow up is not always possible.
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Does $G$ have a clique of size $k$?
How to View CLIQ

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We rephrase that:

I will go to the Zoom whiteboard and do an example, drawing with the mouse.
Wish me luck.
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Let $G = (V, E)$. 

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$G$ has a clique of size $k$ is equivalent to:

There is a 1-1 function $\{1, \ldots, k\} \rightarrow V$ such that for all $1 \leq a, b \leq k$, $(f(a), f(b)) \in E$. 

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**Intent**

\[
x_{ij} = \begin{cases} 
T & \text{if numb } i \text{ maps to vertex } j \\
F & \text{if numb } i \text{ does not map to vertex } j 
\end{cases}
\]
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**Every $i$ maps to at least one $j$**

For $1 \leq i \leq k$

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**Every $i$ maps to at most one $j$**
For $1 \leq i \leq k$, for $1 \leq j_1 < j_2 \leq n$

$$\neg (x_{ij_1} \land x_{ij_2})$$

Note So far all we've used about $G$ is that it has $n$ vertices.
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For $1 \leq i_1 < i_2 \leq k$, for $1 \leq j \leq n$

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Formula: $x_{ij}$ Represent a 1-1 Function

The formula is in different parts to guarantee different things.

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**Note** So far all we’ve used about $G$ is that it has $n$ vertices.
We need that if $i_1$ maps to $j_1$ and $i_2$ maps to $j_2$ then $(j_1, j_2) \in E$. 
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For every $1 \leq i_1 < i_2 \leq k$

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\bigvee_{(j_1, j_2) \in E} x_{i_1j_1} \land x_{i_2j_2}.
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How Big is the Formula

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For $1 \leq i \leq k$: $x_{i1} \vee x_{i2} \vee \cdots \vee x_{in}$. $O(kn)$.

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For every $1 \leq i_1 < i_2 \leq k$, $\bigvee_{(j_1,j_2) \in E} x_{i_1j_1} \land x_{i_2j_2}$. $O(k^2|E|)$
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▶ The formula is of size $O(kn^2) + O(k^2n) + O(k^2|E|)$.  

▶ The construction is easy to do. Yaelle could code this up.

▶ The constants are small.

▶ Usually $k \ll n$ so the real issue is the $n^2$ and the $|E|$. 

Upshot: probably really good on sparse graphs.
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