Closure Properties of P and NP

Exposition by William Gasarch—U of MD

Closure of P

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Closure of P under Union

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  Break string into 1 piece: $\binom{n}{0}$ ways to do this.
  Break string into 2 pieces: $\binom{n}{1}$ ways to do this.
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  So total number of ways to break up the string is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

  What is another name for this?

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B is Bill, **D** is Darling.

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B: No. It means our answers are equal:

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

D: Really!

B: Yes, really!

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Intuition $x_1 \cdots x_i \in L^*$ IFF it can be broken into TWO pieces, the first one in L^*, and the second in L.

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3. Note that we did not include complementation. We'll get to that later.

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Witness $|y| = p_1(|x|) + p_2(|x|) + 1$ is short.

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The following defines $L_1 \cup L_2$ in an NP-way.
$L_1 \cup L_2 = \{x : (\exists y)$
$[$
$|y| = p_1(|x|) + p_2(|x|) + 1 \wedge$
$y = y_1 \$ y_2$ where $|y_1| = p_1(|x|)$ and $|y_2| = p_2(|x|) \wedge$
$(x, y_1) \in B_1 \vee (x, y_2) \in B_2)$
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Witness $|y| = p_1(|x|) + p_2(|x|) + 1$ is short.
Verification $(x, y_1) \in B_1 \vee (x, y_2) \in B_2)$, is quick.

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$$\{x : (\exists z_1, \ldots, z_k, y_1, \ldots, y_k)$$

$$[$$

- $x = z_1 \cdots z_k$
- $(\forall i)[|y_i| = p(|z_i|)]$
- $(\forall i)[(z_i, y_i) \in B]$

$$]\}$$

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Answer **Unknown to Science!**

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It is thought that there is no way for Alice to do this.