Hard Cases for SAT Solvers

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Example

The AND of the following:

1. $x_{11} \lor x_{12}$
2. $x_{21} \lor x_{22}$
3. $x_{31} \lor x_{32}$
4. $\neg x_{11} \lor \neg x_{21}$
5. $\neg x_{11} \lor \neg x_{31}$
6. $\neg x_{21} \lor \neg x_{31}$
7. $\neg x_{12} \lor \neg x_{22}$
8. $\neg x_{12} \lor \neg x_{32}$
9. $\neg x_{22} \lor \neg x_{32}$

This is Pigeonhole Principle: $x_{ij}$ is putting $i$th pigeon in $j$th hole! Can't put 3 pigeons into 2 holes! So Fml is NOT satisfiable.
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PHP: Pigeon Hole Principle

Let \( n < m \). \( n \) is NUMBER OF HOLES, \( m \) is NUMBER OF PIGEONS. \( x_{ij} \) will be thought of as Pigeon \( i \) IS in Hole \( j \).

**Definition**

\( \text{PHP}_n^m \) is the AND of the following:

1. For \( 1 \leq i \leq m \)

\[ x_{i1} \lor x_{i2} \lor \cdots \lor x_{in} \]

(Pigeon \( i \) is in SOME Hole.)

2. For \( 1 \leq i_1 < i_2 \leq m \) and \( 1 \leq j \leq n \)

\[ \neg x_{i1j} \lor \neg x_{i2j} \]

(Hole \( j \) does not have BOTH Pigeon \( i_1 \) and Pigeon \( i_2 \).)

**NOTE:** \( \text{PHP}_n^m \) has \( nm \) **VARS** and \( O(mn^2) \) **CLAUSES** and is NOT satisfiable.
What is Known

1. If $n < m$ then $PHP^{m}_n$ is not satisfiable.
2. The proof of this is by the Pigeon hole principle and not by Truth Table, it was by mathematical reasoning.
3. There is a proof technique called Resolution that is used to show formulas are not satisfiable. It is known that resolution proofs that $PHP^{m}_n$ is not satisfiable are large.
4. Our speculation is that the SAT Solvers we have been studying will take a long time on $PHP^{m}_n$.
5. Try our out SAT solvers on $PHP^{n+1}_n$, $PHP^{n+2}_n$, \ldots and see if it takes a long time. See what happens as the $m$ gets bigger.