BILL AND NATHAN START RECORDING

Review for the Midterm

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7. **Scope of the Exam**

   Short Answer HWs and lectures.

   Long Answer This Presentation.

What We Have Covered: Regular Languages

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5. Non-Regular: ZW Pumping Lemma, Closure properties.

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2. Chomsky Normal Form. Needed to make size comparisons.
3. Closure Properties.
4. Non-CFL's:
   If $L \subseteq a^*$ and not regular, than not CFL.
   If need to keep track of TWO things then NOT CFL.
   E.g., $\{a^n b^n c^n : n \in \mathsf{N}\}$

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   NFA: *e*-transition from one machine to the other.
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5. **Star** What to use?
   NFA: transitions from final to new start/final to start.
   REGEX: By Def.

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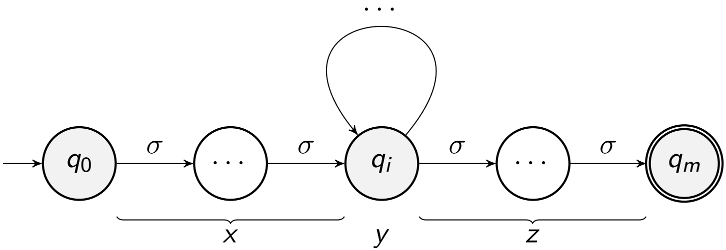
3. For all $i \geq 0$, $xy^iz \in L$.

Proof is picture on the next slide.

Proof by Pictures



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We restate it in the way that we use it.

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3. for all i, $xy^iz \in L$.

We then find some i such that $xy^iz \notin L$ for the contradiction.

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Assume L_1 reg. by PL for long enough string $a^n b^n \in L_1$ there exists x, y, z such that:

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Contradiction since $m_2 \geq 1$.

$L_2 = \{w : \sharp a(w) = \sharp b(w)\}$ is Not Regular

Proof: Same Proof as L_1 not Reg : Still look at $a^m b^m$.
Key Pumping Lemma says for ALL long enough $w \in L$.

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$$(n_1 + n_3) + in_2 \geq x^2 + 2ix + i^2$$

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$$\frac{(n_1 + n_3)}{i} + n_2 \geq i$$

As i increases the LHS decreases and the RHS goes to infinity, so this cannot hold for all i.

$L_8 = \{a^{n_1} b^m c^{n_2} : n_1, n_2 > m\}$ is Not Regular

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Key We are used to thinking of i large. But we can also take $i = 0$, cut out that part of the word. We take $i = 0$ to get

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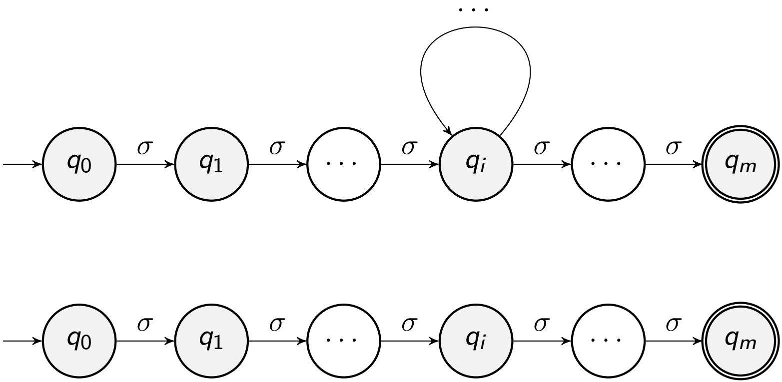
Since $n_2 \ge 1$, we have that $\sharp a(xy^0z) < n \le n - 1 = \sharp b(xy^0z)$.
Hence $xy^0z \notin L_8$.
(There were two other proofs by students: One used that REG closed under PREFIX, and one managed to pump in the middle.)

$i = 0$ Case as a Picture



Answer to SUBSEQ Problem: CFL

If L is CFL than $SUBSEQ(L)$ is CFL.

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If L is CFL than $SUBSEQ(L)$ is CFL. YES.
Let M be a CFL for L in Chomsky Normal Form.

We form a CFL $SUBSEQ(L)$.

For every rule $A \rightarrow \sigma$ we add $A \rightarrow \epsilon$.

Context Free Languages

Definition

A **Context Free Grammar (CFL)** is (V, Σ, P, S)

- V is set of **nonterminals**
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- $P \subseteq V \times (V \cup \Sigma)^*$ are the **productions** or **rules**
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A CFL is in **Chomsky Normal Form CNF)** if all of he productions are either of the form
$A \rightarrow BC$
$A \rightarrow \sigma$ where $\sigma \in \Sigma$
$A \rightarrow e$ (I didn't include it in class, but I am now.)
Note: If G is a CFL hen there exists a CNF CFL that generates it.

Examples of CFL's that are NOT Regular

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To prove it works requires a proof by induction
Not to worry, I will ASSUME you could do such a proof and hence WILL NOT make you.

Examples of Langs with Small CFL's, Large NFA's

$L = \{a^n\}$

- NFA **requires** $\geq n - 2$ states. Lets prove it

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- NFA **requires** $\geq n-2$ states. Lets prove it
  If M is an NFA with $\leq n-2$ states then find a path from the start state to the final state. Let a^m be the shortest string that take you from the start state to the final state. Since the number of states is $\leq n-2$, $m \leq n-2$. So we have a^m accepted when it should not be. Contradiction.
- There is a CNF CFL with $\leq 2\log_2 n$ rules.
  For $n = 2^n$ VERY EASY. If not then have to write n as a sum of powers of 2. Example on next slide.

CNF CFG for $\{a^{10}\}$

$10 = 2^3 + 2^1$

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