The Complexity of Grid Coloring

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1. $G_{n,m}$ is c-colorable if there is a c-coloring of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c-colorable.

Examples

A FAILED 2-Coloring of $G_{4,4}$

R	B	B	R
B	R	R	B
B	B	R	R
R	R	R	B

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R	B	B	R
B	R	R	B
B	B	R	R
R	B	R	B

Example: a 3-Coloring of G(10,10)

Example: A 3-Coloring of $G_{10,10}$

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

It is known that **cannot** 2-color $G_{10,10}$. Hence $\chi(G_{10,10}) = 3$.

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   $G_{n,m}$ is c-colorable iff no element of OBS_c is inside $G_{m,n}$.

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   $$G_{n,m} \text{ is } c\text{-colorable iff no element of } OBS_c \text{ is inside } G_{m,n}.$$

2. We have a proof which shows $|OBS_c| \leq 2c^2$.
3. If OBS_c is known then the set of c-colorable grids is completely characterized.

OBS-2 and OBS-3 Known

We showed

$$OBS_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}$$

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We showed

$$\begin{aligned} OBS_2 &= \{G_{3,7}, G_{5,5}, G_{7,3}\} \\ OBS_3 &= \{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\} \end{aligned}$$

OBS-2 and OBS-3 Known

We showed

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OBS_3 &= \{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}
\end{aligned}
$$

2-colorability table. C for Colorable, U for Uncolorability.

	2	3	4	5	6	7
2	C	C	C	C	C	C
3	C	C	C	C	C	U
4	C	C	C	C	C	U
5	C	C	C	U	U	U
6	C	C	C	U	U	U
7	C	U	U	U	U	U

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3. 2009: Gasarch offered a prize of \$289.00 (17^2) to the first person to email him a 4-coloring of $G_{17,17}$.
4. Brian Hayes, Scientific American Math Editor, popularized the challenge.

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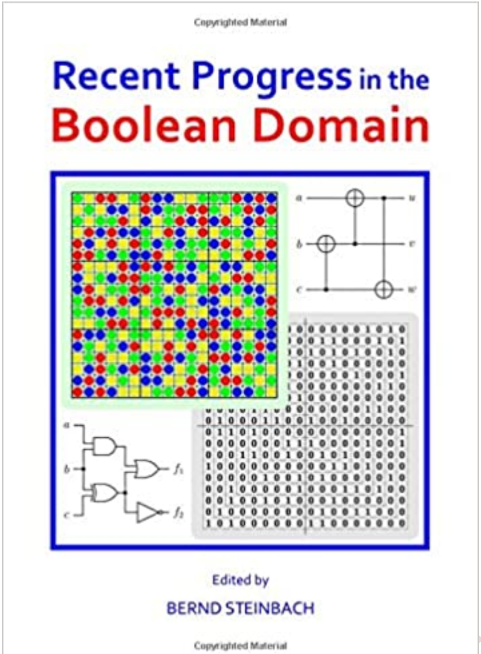
1. Lots of people worked on it.
2. For a while, no progress.
3. In 2012 Bernd Steinbach and Christian Posthoff [SP]. Clever SAT-solver designed for this purpose. Did not generalize.
4. They and others also found colorings that lead to $\mathrm{OBS}_4 = \{$

$$G_{5,41}, G_{6,31}, G_{7,29}, G_{9,25}, G_{18,23}, G_{11,22}, G_{13,21}, G_{17,19},$$
$$G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,18}, G_{22,11}, G_{21,13}, G_{19,17}$$

$\}$

Coloring of $G_{18,18}$ on Book Cover!



Is Grid Coloring Hard?

We view this two ways:

1. Is there an NP-complete problem lurking here somewhere?

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2. Is there a Prop Statement about Grid Coloring whose resolution proof requires exp size?

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Part I of Talk—NP Completeness of GCE

THERE IS AN NP-COMPLETE PROBLEM LURKING!

Grid Coloring Hard!-NP stuff

1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \ldots, c\}$ is a *extendable to a c-coloring* if there is an extension of χ to a total mapping which is a c-coloring of $N \times M$.

Grid Coloring Hard!-NP stuff

1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \ldots, c\}$ is a *extendable to a c-coloring* if there is an extension of χ to a total mapping which is a c-coloring of $N \times M$.
2. $$GCE = \{(N, M, c, \chi) \mid \chi \text{ is extendable}\}.$$

   GCE is NP-complete!

GCE is NP-complete

$\phi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m$ is a 3-CNF formula. We determine N, M, c and a partial c-coloring χ of $N \times M$ such that

$$\phi \in \text{3-SAT iff } (N, M, c, \chi) \in GCE$$

Forcing a Color to Only Appear Once in Main Grid

G								
G								
R		**G**						
G								
G								
G								
G	**G**	**G**	**G**	**G**	**G**	**G**	**G**	**G**

G can only appear once in the main grid, where it is, but what about **R**? (The double lines are not part of the construction. They are there to separate the main grid from the rest.)

Forcing a Color to Only Appear Once in Main Grid

R	G								
R	G								
R	R		G						
R	G								
R	G								
R	G								
R	G	G	G	G	G	G	G	G	G
R	G	R	R	R	R	R	R	R	R

G can only appear once in the main grid, where it is. **R** cannot appear anywhere in the main grid.

Using Variables

D means that the color is some *distinct*, unique color.

	D	D	D	D	D	D	D	D	D	D	D
$\overline{x}_1$		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
$\overline{x}_1$		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
$\overline{x}_1$		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

The labeled $x_1, \overline{x}_1$ are not part of the grid. They are visual aids.

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	D	D	D	D	D	D	D	D	D	D	D
$\overline{x}_1$		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
$\overline{x}_1$		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
$\overline{x}_1$		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

The labeled $x_1, \overline{x}_1$ are not part of the grid. They are visual aids.
First col forced to be T-F-T-F-T-F or F-T-F-T-F-T

Coding a Clause

$C_1 = L_1 \vee L_2 \vee L_3$. Where L_1, L_2, L_3 are literals (vars or their negations).

	$\cdots$	D	T	T
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_1	$\cdots$		D	F
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_2	$\cdots$			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_3	$\cdots$		F	D
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The L_1, L_2, L_3 are not part of the grid. They are visual aids.

Coding a Clause—More Readable

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1		D	F
L_2			
L_3		F	D

One can show that

- If put any of TTT, TTF, TFT, FTT, FFT, FTF, TFF in first column then can extend to full coloring.
- If put FFF in first column then cannot extend to a full coloring.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F		$*$
L_3	T	F	D

The * is forced to be T.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F	$*$	T
L_3	T	F	D

The * is forced to be F.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F	F	T
L_3	T	F	D

Other Assignments

1. We did (F, F, T).
2. (F, T, F), (T, F, F) are similar.
3. (F, T, T), (T, F, T), (T, T, F), (T, T, T) are easier.

Cannot Use (F,F,F)

$C_1 = L_1 \vee L_2 \vee L_3$. Want that (F,F,F) **cannot** be extended to a coloring.

	D	T	T
L_1	F	D	F
L_2	F	$*$	$*$
L_3	F	F	D

The *'s are forced to be T.

Cannot Use (F,F,F)

	D	**T**	**T**
L_1	F	D	F
L_2	F	**T**	**T**
L_3	F	F	D

There is a mono rectangle of T's. **Not** a valid coloring!

Put it all Together

Do the above for all variables and all clauses to obtain the result that GRID EXT is NP-complete!

Big Example

$$(x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

												C_1	C_1	C_2	C_2	C_3	C_3
	D	D	D	D	D	D	D	D	D	D	D	T	T	T	T	T	T
$\overline{x}_4$		D	D	D	D	D	D	D	D	T	F	D	D	D	D	D	F
x_4		D	D	D	D	D	D	D	D	T	F	D	D	D	F	D	D
$\overline{x}_3$		D	D	D	D	D	D	T	F	D	D	D	D	D	D	D	D
x_3		D	D	D	D	T	F	T	F	D	D	D	D			D	D
$\overline{x}_3$		D	D	D	D	T	F	D	D	D	D	D	F	D	D		
$\overline{x}_2$		D	D	T	F	D	D	D	D	D	D	D	D	F	D	D	D
x_2		D	D	T	F	D	D	D	D	D	D			D	D	D	D
$\overline{x}_1$		T	F	D	D	D	D	D	D	D	D	D	D	D	D	F	D
x_1		T	F	D	D	D	D	D	D	D	D	F	D	D	D	D	D

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1. **Maybe Not** GCE is Fixed Parameter Tractable. For fixed c GCE_c is in time $O(N^2M^2 + 2^{O(c^4)})$. But for $c = 4$ this is huge!
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2. **Maybe Yes** Gives a sense that brute force search is needed- not shortcuts.
3. **Maybe Not** Our result says nothing about the case where the grid is originally all blank.
4. **Maybe Yes** GCE problem should be easier than starting with all blanks.

Key to $O(N^2M^2 + 2^{O(c^4)})$ Result

Lemma Let χ be a partial c-coloring of $G_{n,m}$. Let U be the uncolored grid points. Let $|U| = u$. There is an algorithm that will determine if χ can be extended to a full c-coloring that runs in time $O(cnm2^{2u}) = 2^{O(nm)}$.
Set Up the Algorithm For $S \subseteq U$ and $1 \le i \le c$ let

$$f(S,i) = \begin{cases} \textbf{Yes} & \text{if } \chi \text{ can be extended to } S \text{ using colors } \{1,\ldots,i\}; \\ \textbf{No} & \text{if not.} \end{cases}$$

For $S \subseteq U$ and $1 \le i \le c$ use Dynamic Programming to compute $f(S,i)$. $f(U,c)$ is your answer.
End of Set Up of Algorithm

Computing $f(S, i)$

Assume that $(\forall S', |S'| < |S|)(\forall 1 \le i \le c)[f(S', i)$ is known].

1. For all 1-colorable $T \subseteq S$ do the following
   1.1 If $f(S - T, i) = NO$ then $f(S, i) = NO$ and STOP.
   1.2 If $f(S - T, i - 1) = YES$ then determine if coloring T with i works. If yes then $f(S, i) = YES$ and STOP. Note that this takes $O(nm)$.
2. We know that for all 1-colorable $T \subseteq S$ $f(S - T, i) = YES$ and either
   (1) $f(S - T, i - 1) = NO$ or
   (2) $f(S - T, i - 1) = YES$ and coloring T with i bad.
   In all cases $f(S, i) = NO$.

Open Questions

1. Improve Fixed Parameter Tractable algorithm.
2. NPC results for mono squares? Other shapes?
3. Show that

$$\{(n, m, c) : G_{n,m} \text{ is } c\text{-colorable } \}$$

   is hard.
   - If n, m in unary then sparse set, not NPC—New framework for hardness needed.
   - If n, m binary then not in NP. Could try to prove NEXP-complete. But the difficulty of the problem is not with the grid being large, but with the number-of-possibilities being large.

Part II of Talk—Lower Bounds on Tree Resolution

YOU SAY YOU WANT A RESOLUTION!

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YOU SAY YOU WANT A RESOLUTION!

If you write a good parody of the Beatles **You say you want a revolution** about resolution theorem proving, I will treat you to lunch at **The Food Factory**, if they open up again.

Resolution

Def Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula.
A **Resolution Proof** of $\varphi \notin \mathrm{SAT}$ is a sequence of clauses such that on each line you have either

1. One of the C's in φ (called an **Axiom**).
2. $A \vee B$ if $A \vee x$ and $B \vee \neg x$ were on prior lines. Variable that is **resolved on** is x.
3. The last line has the empty clause.

Example

$\varphi = x_1 \wedge x_2 \wedge (\neg x_1 \vee \neg x_2)$

1. x_1 **Axiom**
2. $\neg x_1 \vee \neg x_2$ **AXIOM**
3. $\neg x_2$ (From lines 1,2, resolve on x_1.)
4. x_2 **Axiom**
5. $\emptyset$ (From lines 3,4, resolve on x_2.)

Resolution is Complete

Def Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula on n variables.

1. If exists a Res Proof of $\varphi \notin SAT$ then $\varphi \notin \mathrm{SAT}$.
   Proof Any assignment that satisfies φ satisfies any node of the Res Proof including the last node $\emptyset$.
2. If $\varphi \notin \mathrm{SAT}$ then exists a Res Proof of $\varphi \notin \mathrm{SAT}$ of size $2^{O(n)}$.
   Proof Form a Decision Tree that has at every node on level i the variable x_i. Right$=T$ and Left$=F$. A leaf is the first clause that is false with that assignment. **Turn Decision Tree upside down! View nodes as which var to resolve on! This will be Res Proof!** (It will even be Tree Res Proof.)

Another Example

The **and** of the following:

1. For $i, j \in \{1, \ldots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

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$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$

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3. For $i, j, i', j' \in \{1, \ldots, 5\}$
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Interpretation There is no mono 2-rectangle.

We interpret this statement as saying

There is a 2-coloring of $G_{5,5}$.

This statement is known to be false.

$\mathrm{GRID}(n, m, c)$

Def Let $n, m, c \in \mathbb{N}$. $\mathrm{GRID}(n, m, c)$ is the **and** of the following:

1. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$,

$$x_{ij1} \vee x_{ij2} \vee \cdots \vee x_{ijc}$$

   Interpretation (i, j) is colored either 1 or $\cdots$ or c.

2. For $i, i' \in \{1, \ldots, n\}$, $j, j' \in \{1, \ldots, m\}$, $k \in \{1, \ldots, c\}$,

$$\neg x_{ijk} \vee \neg x_{i'jk} \vee \neg x_{ij'k} \vee \neg x_{i'j'k}$$

   Interpretation There is no mono rectangle.

We interpret this statement as saying

There is a c-coloring of $G_{n,m}$.

Note $\mathrm{GRID}(n, m, c)$ has nmc **VARS** and $O(cn^2m^2)$ **CLAUSES.**

$\mathrm{GRID}(n, m, c)$—How to View Assignments

Given an assignment:

1. For all $i \in [n]$ and $j \in [m]$ let k be the **least** number such that $x_{ijk} = T$. View this as saying that (i, j) is colored k.
2. If there is NO such number then (i, j) is not colored and this assignment makes $\mathrm{GRID}(n, m, c)$ **false**.

Hence we view assignments as attempted colorings of the grid where some points are not colored.

Two Ways to Invalidate $\mathrm{GRID}(n, m, c)$

1. There is a mono rectangle.
2. There is some point that is not colored: there is some i, j such that all x_{ijk} are **false**.

Tree Resolution Proofs

Def A **Tree Res Proof** is a Res Proof where the underlying graph is a tree. Note that if you remove the bottom node that is labeled $\emptyset$ then the Tree Res Proof is cut into two **disjoint** parts.
Known If $\varphi \notin \mathrm{SAT}$ and φ has v variables then there is a Tree Res Proof of φ of size $2^{O(v)}$.

Our Goal

Assume that there is no c-coloring of $G_{n,m}$.

1. $\mathrm{GRID}(n, m, c)$ has a size $2^{O(cnm)}$ Tree Res Proof.
2. We show $2^{\Omega(c)}$ size is **required**. This is our point!
3. The lower bound is **independent** of n, m.

Interesting Examples

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1. We showed that $G_{2c^2-c,,2c}$ is not c-colorable. Hence

$$\mathrm{GRID}(2c^2 - c, 2c)$$

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   has $O(c^3)$ vars, $O(c^6)$ clauses but $2^{\Omega(c)}$ Tree Res proof.

2. Easy to show G_{c^3,c^3} is not c-colorable.

$$\mathrm{GRID}(c^3, c^3, c)$$

   has $O(c^7)$ vars, $O(c^{13})$ clauses and $2^{\Omega(c)}$ Tree Res proof.

These are poly-in-c formulas that **require** $2^{\Omega(c)}$ Tree Res proofs.

The Prover-Delayer Game

(Due to Pudlak and Impagliazzo [PI].) Parameters of the game: $p \in \mathsf{R}^+$,

$$\varphi = C_1 \wedge \cdots \wedge C_L \notin \mathrm{SAT}.$$

Do the following until a clause is proven false:

1. **PROVER** picks a variable x that was not already picked.
2. **DEL** either
   2.1 Sets x to F or T, OR
   2.2 Defers to **PROVER** who then sets x to T or F while **DEL** gets a point.

At end if **DEL** has $\geq p$ pts then he **wins**; else **PROVER** **wins**.

Convention

We assume that **PROVER** and **DEL** play perfectly.

1. **PROVER** *wins* means **PROVER** *has a winning strategy*.
2. **DEL** *wins* means **DEL** *has a winning strategy*.

Prover-Delayer Game and Tree Res Proofs

Lemma Let $p \in \mathrm{R}^+$, $\varphi \notin \mathrm{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins.

Prover-Delayer Game and Tree Res Proofs

Lemma Let $p \in \mathrm{R}^+$, $\varphi \notin \mathrm{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins. **Pf** **PROVER** Strategy:

1. Initially T is res tree of size $< 2^p$ and **DEL** has 0 pts.
2. **PROVER** picks x, the **last** var **resolved on**.
3. If **DEL** sets x then **DEL** gets no pts.
4. If **DEL** defers then **PROVER** sets T or F—**whichever yields a smaller tree.** **Note** One of the trees will be of size $< 2^{p-1}$. **DEL** gets 1 point.
5. Repeat: after ith stage will always have T of size $< 2^{p-i}$, and **DEL** has $\leq i$ pts.

Contrapositive is Awesome!

Recall:

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Lemma Let $p \in \mathrm{R}^{+}$, $\varphi \notin \mathrm{SAT}$. If φ has a Tree Res Proof of size $< 2^{p}$ then **PROVER** wins.

Contrapositive
Lemma Let $p \in \mathrm{R}^{+}$, $\varphi \notin \mathrm{SAT}$. If **DEL** wins then **EVERY** Tree Res Proof for φ has size $\geq 2^{p}$. **Plan** Get **awesome** strategy for **DEL** when $\varphi = \mathrm{GRID}(n, m, c)$.

$\mathrm{GRID}(n, m, c)$ Requires Exp Tree Res Proofs

Thm Let n, m, c be such that $G_{n,m}$ is not c-colorable. Let $c \geq 2$. Any tree resolution proof of $\mathrm{GRID}(n, m, c) \notin \mathrm{SAT}$ requires size $2^{0.5c}$.

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Pf Parameters: $p = 0.5c$, $\varphi = \mathrm{GRID}(n, m, c)$.

Del Strategy

Assume x_{ijk} was chosen by the **PROVER**.

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Assume x_{ijk} was chosen by the **PROVER**.

1. If setting $x_{ijk} = T$ creates a mono rect (of color k) then **DEL** **does not** let this happen— he sets x_{ijk} to F.
2. If none of the x_{ij*} are T and $\geq \frac{c}{2}$ of the x_{ij*} are F via **PROVER** then **DEL** sets x_{ijk} to T.
3. In all other cases the **DEL** defers to the **PROVER**.

Case 1: Prover Set c/2 Vars to F

Game ends when there is some i, j such that

$$x_{ij1} = x_{ij2} = \cdots = x_{ijc} = F.$$

Who set those variables to F?

Case 1 At least $\frac{c}{2}$ set F by Prover. Then **DEL** gets at least

$$0.5c \text{ pts.}$$

Case 2: Del Set c/2 Vars to F

$x_{ij1} = x_{ij2} = \cdots = x_{ijc} = F$. Who set those vars to F?
Case 2 At least $\frac{c}{2}$ set F by **DEL**. Assume they are
$x_{ij1}, x_{ij2}, \ldots, x_{ijc/2}$.

Case 2: Del Set c/2 Vars to F

$x_{ij1} = x_{ij2} = \cdots = x_{ijc} = F$. Who set those vars to F?
Case 2 At least $\frac{c}{2}$ set F by **DEL**. Assume they are $x_{ij1}, x_{ij2}, \ldots, x_{ijc/2}$.

- x_{ij1} set to F by **DEL**. Why? There exists i', j' such that $x_{i'j1}$, $x_{ij'1}$, $x_{i'j'1}$ all set to T. (Do not know by who.)

Case 2: Del Set c/2 Vars to F

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Case 2 At least $\frac{c}{2}$ set F by **DEL**. Assume they are $x_{ij1}, x_{ij2}, \ldots, x_{ijc/2}$.

- x_{ij1} set to F by **DEL**. Why? There exists i', j' such that $x_{i'j1}$, $x_{ij'1}$, $x_{i'j'1}$ all set to T. (Do not know by who.)
- x_{ij2} set to F by **DEL**. Why? There exists i'', j'' such that $x_{i''j2}$, $x_{ij''2}$, $x_{i''j''2}$ all set to T. (Do not know by who.)

Case 2: Del Set c/2 Vars to F

$x_{ij1} = x_{ij2} = \cdots = x_{ijc} = F$. Who set those vars to F?
Case 2 At least $\frac{c}{2}$ set F by **DEL**. Assume they are $x_{ij1}, x_{ij2}, \ldots, x_{ijc/2}$.

- x_{ij1} set to F by **DEL**. Why? There exists i', j' such that $x_{i'j1}$, $x_{ij'1}$, $x_{i'j'1}$ all set to T. (Do not know by who.)
- x_{ij2} set to F by **DEL**. Why? There exists i'', j'' such that $x_{i''j2}$, $x_{ij''2}$, $x_{i''j''2}$ all set to T. (Do not know by who.)
- etc.

For every k such that x_{ijk} is set to F by **DEL** there exists **three** vars of form x_{**k} set to T.
Key All these 3-sets are **disjoint**, so at least $3c/2$ vars set T (by who?).

Case 2a: Prover Set 3c/2 Vars to T

Key At least $3c/2$ vars set T (by who?).
Case 2a **PROVER** set $\geq \frac{3c}{4}$ to T. **DEL** gets at least

$$0.75c = 0.75c \text{ pts.}$$

Case 2b: Del Set 3c/2 Vars To T

Case 2b **DEL** set $\geq \frac{3c}{4}$ to T.
DEL set x_{ijk} to T:

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Case 2b **DEL** set $\geq \frac{3c}{4}$ to T.

DEL set x_{ijk} to T:

- At time there are $c/2$ k' such that **PROVER** set $x_{ijk'}$ to F.

Case 2b: Del Set 3c/2 Vars To T

Case 2b **DEL** set $\geq \frac{3c}{4}$ to T.
DEL set x_{ijk} to T:

- At time there are $c/2$ k' such that **PROVER** set $x_{ijk'}$ to F.
- **DEL** will **never** set an x_{ij*} to T again! **never**!!

Every x_{ijk} set T by **DEL** implies that $c/2$ vars set F by **PROVER**, and these sets of $c/2$ vars are disjoint.
Upshot **PROVER** had set $\frac{3c}{4} \times \frac{c}{2}$ to F. **DEL** gets at least

$$0.375c^2 = 0.375c^2 \text{ pts.}$$

Final Analysis

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- **Case 1** **DEL** gets at least $0.5c$ pts.

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- **Case 1 DEL** gets at least $0.5c$ pts.
- **Case 2a DEL** gets at least $0.75c$ pts.
- **Case 2b DEL** gets at least $0.375c^2$ pts.

Upshot For $c \geq 2$ **DEL** gets at least $0.5c$ pts.
Punchline By **Lemma** any Tree Res Proof has size $\geq 2^{0.5c}$.

Optimize

1. In construction use cutoff of $c/2$ for when **DEL** sets x_{ijk} to T. Choose fraction **carefully**.
2. In analysis we twice do a half-half cutoff. Choose fractions **carefully**!
3. Use asymmetric **PROVER-DEL** game (next slide) and choose a, b **carefully**!

Thm Let n, m, c be such that $G_{n,m}$ is not c-colorable. Let $c \geq 9288$. Any tree resolution proof of $\mathrm{GRID}(n, m, c) \notin \mathrm{SAT}$ requires size $2^{0.836c}$.

Asymmetric Prover-Delayer Game

(Due to Beyersdorr, Galesi, Lauria [BGL].) Parameters of the game: $a, b \in (1, \infty)$, with $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathrm{R}^+$,

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      2.2.2 If **PROVER** sets $x = T$ then **DEL** gets $\lg b$ pts.

At end if **DEL** has $\geq p$ pts then he **wins**; else **PROVER** **wins**.

Other Shapes

What is special about rectangles? **Nothing**! **Def** (Informally) Let S be a set of at least 2 grid points. Let $\mathrm{GRID}(n, m, c; S)$ be the prop statement that there is a c-coloring of $G_{n,m}$ with no mono configuration that is "like S".

Thm (Informally) Let S be a set of at least 2 grid points. Let n, m, c be such that $\mathrm{GRID}(n, m, c; S) \notin \mathrm{SAT}$. Any tree resolution proof of $\mathrm{GRID}(n, m, c; S) \notin SAT$ requires size $2^{\Omega(c)}$.

Open Questions

1. Want matching upper bounds for Tree Res Proofs of $\mathrm{GRID}(n, m, c) \notin \mathrm{SAT}$.
2. Want lower bounds on Gen Res Proofs of $\mathrm{GRID}(n, m, c) \notin \mathrm{SAT}$.
3. Want lower bounds on in other proof systems $\mathrm{GRID}(n, m, c) \notin \mathrm{SAT}$.
4. Upper and lower bounds for $\mathrm{GRID}(n, m, c; S)$ for various S in various proof systems.

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