

# Concrete Time Hierarchy Thm

Exposition by William Gasarch—U of MD

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1. If  $x \in A$  then  $J(x)$  outputs YES.
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2.  $A \notin \text{DTIME}(n^3)$ .

# ASCII Table

Hex	Dec	Char	Hex	Dec	Char	Hex	Dec	Char	Hex	Dec	Char
0x00	0	<b>NULL</b> null	0x20	32	<b>Space</b>	0x40	64	<b>@</b>	0x60	96	<b>~</b>
0x01	1	<b>SOH</b> Start of heading	0x21	33	<b>!</b>	0x41	65	<b>A</b>	0x61	97	<b>a</b>
0x02	2	<b>STX</b> Start of text	0x22	34	<b>"</b>	0x42	66	<b>B</b>	0x62	98	<b>b</b>
0x03	3	<b>ETX</b> End of text	0x23	35	<b>#</b>	0x43	67	<b>C</b>	0x63	99	<b>c</b>
0x04	4	<b>EOT</b> End of transmission	0x24	36	<b>\$</b>	0x44	68	<b>D</b>	0x64	100	<b>d</b>
0x05	5	<b>ENQ</b> Enquiry	0x25	37	<b>%</b>	0x45	69	<b>E</b>	0x65	101	<b>e</b>
0x06	6	<b>ACK</b> Acknowledge	0x26	38	<b>&amp;</b>	0x46	70	<b>F</b>	0x66	102	<b>f</b>
0x07	7	<b>BELL</b> Bell	0x27	39	<b>'</b>	0x47	71	<b>G</b>	0x67	103	<b>g</b>
0x08	8	<b>BS</b> Backspace	0x28	40	<b>(</b>	0x48	72	<b>H</b>	0x68	104	<b>h</b>
0x09	9	<b>TAB</b> Horizontal tab	0x29	41	<b>)</b>	0x49	73	<b>I</b>	0x69	105	<b>i</b>
0x0A	10	<b>LF</b> New line	0x2A	42	<b>*</b>	0x4A	74	<b>J</b>	0x6A	106	<b>j</b>
0x0B	11	<b>VT</b> Vertical tab	0x2B	43	<b>+</b>	0x4B	75	<b>K</b>	0x6B	107	<b>k</b>
0x0C	12	<b>FF</b> Form Feed	0x2C	44	<b>,</b>	0x4C	76	<b>L</b>	0x6C	108	<b>l</b>
0x0D	13	<b>CR</b> Carriage return	0x2D	45	<b>-</b>	0x4D	77	<b>M</b>	0x6D	109	<b>m</b>
0x0E	14	<b>SO</b> Shift out	0x2E	46	<b>.</b>	0x4E	78	<b>N</b>	0x6E	110	<b>n</b>
0x0F	15	<b>SI</b> Shift in	0x2F	47	<b>/</b>	0x4F	79	<b>O</b>	0x6F	111	<b>o</b>
0x10	16	<b>DLE</b> Data link escape	0x30	48	<b>0</b>	0x50	80	<b>P</b>	0x70	112	<b>p</b>
0x11	17	<b>DC1</b> Device control 1	0x31	49	<b>1</b>	0x51	81	<b>Q</b>	0x71	113	<b>q</b>
0x12	18	<b>DC2</b> Device control 2	0x32	50	<b>2</b>	0x52	82	<b>R</b>	0x72	114	<b>r</b>
0x13	19	<b>DC3</b> Device control 3	0x33	51	<b>3</b>	0x53	83	<b>S</b>	0x73	115	<b>s</b>
0x14	20	<b>DC4</b> Device control 4	0x34	52	<b>4</b>	0x54	84	<b>T</b>	0x74	116	<b>t</b>
0x15	21	<b>NAK</b> Negative ack	0x35	53	<b>5</b>	0x55	85	<b>U</b>	0x75	117	<b>u</b>
0x16	22	<b>SYN</b> Synchronous idle	0x36	54	<b>6</b>	0x56	86	<b>V</b>	0x76	118	<b>v</b>
0x17	23	<b>ETB</b> End transmission block	0x37	55	<b>7</b>	0x57	87	<b>W</b>	0x77	119	<b>w</b>
0x18	24	<b>CAN</b> Cancel	0x38	56	<b>8</b>	0x58	88	<b>X</b>	0x78	120	<b>x</b>
0x19	25	<b>EM</b> End of medium	0x39	57	<b>9</b>	0x59	89	<b>Y</b>	0x79	121	<b>y</b>
0x1A	26	<b>SUB</b> Substitute	0x3A	58	<b>:</b>	0x5A	90	<b>Z</b>	0x7A	122	<b>z</b>
0x1B	27	<b>FSC</b> Escape	0x3B	59	<b>;</b>	0x5B	91	<b>[</b>	0x7B	123	<b>{</b>
0x1C	28	<b>FS</b> File separator	0x3C	60	<b>&lt;</b>	0x5C	92	<b>\</b>	0x7C	124	<b> </b>
0x1D	29	<b>GS</b> Group separator	0x3D	61	<b>=</b>	0x5D	93	<b>]</b>	0x7D	125	<b>}</b>
0x1E	30	<b>RS</b> Record separator	0x3E	62	<b>&gt;</b>	0x5E	94	<b>^</b>	0x7E	126	<b>~</b>
0x1F	31	<b>US</b> Unit separator	0x3F	63	<b>?</b>	0x5F	95	<b>_</b>	0x7F	127	<b>DEL</b>

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- ▶ I won't bother with the rest. See table.

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So this piece of code maps to 120,061,120,043,049,050

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$J'_1, J'_2, \dots, \dots$  is the list of all  $n^3$ -time Java Programs .

**Upshot** If  $A \in \text{DTIME}(n^3)$  then there exists  $i$  such that  $J'_i$  recognizes  $A$ .

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**Proof** Let  $A$  be decided by the following program

1. Input( $x$ ). If  $x \notin 0^*$  output NO and stop. Otherwise  $x = 0^n$ .
2. Run  $J'_n(0^n)$ .
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- 1) This is clearly a program that recognizes  $A$ .  
2) Proof that  $A \notin \text{DTIME}(n^3)$  on next slide.

# $A \notin \text{DTIME}(n^3)$

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$\vdots$

$J'_n$  cannot recognize  $A$ :  $J'_n(0^n)$  and  $A(0^n)$  DIFFER.

So NO  $J'_n$  recognizes  $A$ . Hence  $A \notin \text{DTIME}(n^3)$ .