BILL AND NATHAN, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

TSP cannot be Approximated Unless P=NP

TSP

Notation

In this slide packet G is always a weighted graph with natural number weights

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But what about **approximating it**? Need to define this carefully.

An α-Approx For TSP

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Def Let $\alpha > 1$. An **α-approx for TSP** is a poly time algorithm that, on input G, returns a cycle that is $\leq \alpha\mathrm{OPT}(G)$.

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4. Arora and Mitchell actually have an algorithm that works on n points in R^d that runs in time $O(n(\log n)^{O(\sqrt{d}/\epsilon)^{d-1}})$.

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We show that, for all α, TSP does not have an α-approx. (unless $\mathrm{P} = \mathrm{NP}$).

TSP Does Not have an α-Approx

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Comment
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Need to set B such that $\alpha n < B$. $B = n^2$ will suffice.

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Case 1: If $M(G') \le \alpha n$ then output YES.
Case 2: If $M(G') \ge B$ then output NO.

We can Do Better

We showed:
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If you look at the proof more carefully you can prove this:
Thm Let $\alpha(n)$ be a polynomial. If there is an $\alpha(n)$-approx for TSP then P=NP.

What About Metric TSP?

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1. There are math reasons to think the answer is $\frac{4}{3} \sim 1.33$.
2. There is a Neural Net which seems to obtain $\frac{5}{4} \sim 1.25$.