A Sane Proof that $COL_4 \le COL_3$

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COL_k and Reductions

1. $COL_k = \{G \mid\ G \text{ is } k\text{-colorable } \}$
2. $COL_2 \in P$
3. COL_3 is NP-complete (Stockmeyer)
4. EASY to prove that $COL_3 \le COL_4$:

1. Input $G = (V, E)$.
2. Output $G' = (V \cup \{v_{\mathrm{new}}\}, E \cup \{(v, v_{\mathrm{new}}) \mid v \in V\})$
3. Intuition: Add v_{new} and connect to all vertices.

I Ask My Students...

Bill to Class: I just proved $\mathrm{COL}_3 \le \mathrm{COL}_4$.
Is $\mathrm{COL}_4 \le \mathrm{COL}_3$? VOTE:

1. YES $\mathrm{COL}_4 \le \mathrm{COL}_3$.
2. NO $\mathrm{COL}_4 \not\le \mathrm{COL}_3$.
3. UNKNOWN TO SCIENCE!!

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The class results:
YES: 3, NO: 30, UNKNOWN TO SCIENCE: 7

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Class is **WRONG**. Answer is YES:

$$\mathrm{COL}_4 \le \mathrm{SAT} \le \mathrm{COL}_3$$

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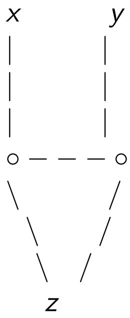
Class is **RIGHT**. This reduction is INSANE!

A GADGET



$\mathrm{GAD}(x_1, \ldots, x_k, z)$ is

$$\mathrm{GAD}(x_1, x_2, y_1)$$
$$\mathrm{GAD}(y_1, x_3, y_2),\ \mathrm{GAD}(y_2, x_4, y_3),\ \ldots,\ \mathrm{GAD}(y_{k-3}, x_{k-1}, y_{k-2}),$$
$$\mathrm{GAD}(y_{k-2}, x_k, z).$$

LEMMA: If $\mathrm{GAD}(x_1, x_2, \ldots, x_k, z)$ is three colored and $x_1, \ldots, x_k$ get the same color, then z also gets that color.

$\mathrm{COL}_4 \le \mathrm{COL}_3$ by a simple reduction (Gasarch)

INPUT(G).
OUTPUT:

1. Vertices T, F, R form a triangle.
2. For $1 \le i \le n$ and $1 \le j \le k$ vertex v_{ij}. All of these will be connected by an edge to vertex R. OUT INTENT:
   - v_{ij} is colored T means that vertex v_i in G is colored j;
   - v_{ij} is colored F means that vertex v_i in G is not colored j.
3. **For all i:** *At least one of $v_{i1}, \ldots, v_{in}$ is colored T*: $\mathrm{GAD}(v_{i1}, \ldots, v_{in}, T)$.
4. **For all i:** *At most one of $v_{i1}, \ldots, v_{in}$ is colored T*: **for all $j_1 < j_2$** $\mathrm{GAD}(v_{ij_1}, v_{ij_2}, F)$.
5. FOR ALL edges (v_i, v_j) in the original graph: $\mathrm{GAD}(v_{i1}, v_{j1}, F)$, $\mathrm{GAD}(v_{i2}, v_{j2}, F)$,. . ., $\mathrm{GAD}(v_{ik}, v_{jk}, F)$.

Clearly G is k-colorable iff G' is 3-colorable.

Open Question

Our reduction takes a graph on n vertices and e edges and produces a graph on $O(n+e)$ vertices and $O(n+e)$ edges.

Can this be improved?

Can this be improved in a way that is not INSANE.