HW 11 CMSC 456. MORALLY DUE Dec 3
NOTE- THE HW IS FIVE LONG

1. (0 points) READ the syllabus- Content and Policy. What is your name? Write it clearly. What is the day of the final? READ the slides and notes on Perfect and Comp Secrecy.

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2. (30 points) In this problem we will guide you through a proof that a bad random string generator will result in a 1-time pad that is NOT secure in that Eve has prob $> \frac{1}{2}$ of winning the security game.

   Alice and Bob are using a 1-time pad. But their random bit generator is terrible! It outputs 0^n with probability $\frac{1}{2} + \frac{1}{2^{n+1}}$, and every other string of length n with probability $\frac{1}{2^{n+1}}$. Answer the following questions which will lead up to a proof that Alice and Bob's 1-time pad leads to an insecure cipher. You can assume n is odd and large.

   (a) (0 points) Eve picks $m_0 = 0^n$ and $m_1 = 1^n$.

   (b) (3 points) What is $\Pr(m = m_0)$? $\Pr(m = m_1)$?

   (c) (0 points) Recall that Alice picks $m \in \{m_0, m_1\}$ and then generates the key k (very badly!) and sends Eve $c = m \oplus k$.

   (d) (0 points) Let MAJ0 be the event: *c is over half 0's.* Let MAJ1 be the event: *c is over half 1's.*. We will take n odd so that either MAJ0 or MAJ1 occurs.

   (e) (3 points) What is $\Pr(MAJ0|m = m_0)$? (Do not use the definition of Cond Prob- use instead that IF $m = m_0$, what is the prob that the key k is such that $c = m \oplus k$ has majority 0's.) (You can approximate by taking $\frac{1}{2^{n+1}}$ to be 0. We are assuming n is large.)

   (f) (3 points) What is $\Pr(MAJ1|m = m_1)$? (Do not use the definition of Cond Prob- use instead that IF $m = m_1$, what is the prob that the key k is such that $c = m \oplus k$ has majority 1's.) (You can approximate by taking $\frac{1}{2^{n+1}}$ to be 0. We are assuming n is large.)

   (g) (3 points) What is $\Pr(MAJ0|m = m_1)$? (Do not use the definition of Cond Prob- use instead that IF $m = m_1$, what is the prob that the key k is such that $c = m \oplus k$ has majority 0's.) (You can approximate by taking $\frac{1}{2^{n+1}}$ to be 0. We are assuming n is large.)

   (h) (3 points) What is $\Pr(MAJ1|m = m_0)$? (Do not use the definition of Cond Prob- use instead that IF $m = m_0$, what is the prob that the key k is such that $c = m \oplus k$ has majority 1's.) (You an approx by taking $\frac{1}{2^{n+1}}$ to be 0. We are assuming n is large.)

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(i) (3 points) What is $\Pr(MAJ0)$? (Hint: its

$$\Pr(MAJ0|m = m_0)\Pr(m = m_0) + \Pr(MAJ0|m = m_1)\Pr(m = m_1)$$

and you have all of those parts.)

(j) (3 points) What is $\Pr(MAJ1)$? (Hint: its

$$\Pr(MAJ1|m = m_0)\Pr(m = m_0) + \Pr(MAJ1|m = m_1)\Pr(m = m_1)$$

and you have all of those parts.)

(k) (3 points) What is $\Pr(m = m_0|MAJ0)$ (Hint: Use Bayes's theorem)

(l) (3 points) What is $\Pr(m = m_1|MAJ1)$ (Hint: Use Bayes's theorem)

(m) (3 points) Show that Eve has a winning strategy. Describe the strategy and use the parts above to show it has prob $> \frac{1}{2}$ of winning. What is the prob of Eve winning?

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3. (30 points) In this problem we will make what we said about the Randomized Shift rigorous lay the groundwork for being able to apply the technique elsewhere.

   (a) (10 points) (Look up THE BIRTHDAY PARADOX on the web though you will need to adjust it some.) Find a number a such that, for large N, $a\sqrt{N}$ elements from $\{1, \ldots, N\}$ with replacement then the probability that two are the same is $\geq \frac{3}{4}$ (the traditional birthday Paradox is $\frac{1}{2}$ so you will need to adjust this.) You have to hand in a self-contained account, you can't say *see website BLAH*.

   (b) (10 points) Assume you are doing randomized shift with an alphabet of size N. Show that the randomized shift is not computationally secure by giving a strategy in the comp sec game where Eve wins with prob much bigger than $\frac{1}{2}$. (You may use part 1 above.)

   (c) (10 points) Assume the alphabet size N is prime. Hence the number of (a, b) such that $ax + b$ is a valid Affine Cipher is N^2 (we will not let $b = 0$). Recall the RANDOMIZED AFFINE CIPHER:

      i. Alice and Bob both have the key which is a function $f : \{1, \ldots, N^2\} \to \{1, \ldots, N\} \times \{1, \ldots, N\}$.

      ii. For Alice to send Bob a message $\sigma_1, \sigma_2, \ldots, \sigma_L$ he (1) generates RANDOM $r_1, \ldots, r_L \in \{1, \ldots, N^2\}$, (2) for $1 \leq i \leq L$ Alice finds $f(r_i) = (a_i, b_i)$. (3) sends

      $$(r_1, a_1\sigma_1 + b_1), (r_2, a_2\sigma_2 + b_2), \ldots, (r_L, a_L\sigma_L + b_L)$$

      iii. We leave it to you for how Bob decodes, but note that since he has r_i's he can find a_i's and b_i's.

   Show that the randomized affine is not computationally secure by giving a strategy in the comp sec game where Eve wins with prob much bigger than $\frac{1}{2}$. (You may need to reason a bit informally towards the end of the proof.)

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4. (40 points) State two facts you learned from Lloyd's talk on the NSA.