Msg Auth Codes (MAC), Hashing, Digital Signatures
Authentication

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Terminology: Security is not the right term. Non-forgemability is. We still use the term Security Parameter.
**Formal Def of MAC**

**Def:** A MAC is $\Pi = (GEN, MAC, V)$ where:

1. $GEN(1^n)$ is a uniform $k \in \{0, 1\}^n$.
2. Given key $k$ and msg $m$, $MAC_k(m) = t$, a tag. $MAC_k$ is PPT.
3. $V_k(m, t) = 1$ if $MAC_k(m) = t$, 0 otherwise.

**How to Use:** Alice and Bob have $\Pi = (GEN, MAC, V)$

1. Alice generates $k$ via $GEN$ and sends it to Bob privately.
2. To send $m \in \{0, 1\}^*$ to Alice, Bob computes $t = MAC_k(m)$ and sends $(m, t)$.
3. Alice authenticates that its from Bob iff $V_k(m, t) = 1$.

**Note:** We often restrict to $m \in \{0, 1\}^{p(n)}$, $p$ poly.
Example of a MAC

1. \( k \in \{0, \ldots, p - 1\} \) unif.
2. \( MAC_k(m) = m + k \).
3. \( V_k(m, t) = 1 \) if \( t = m + k \)

**Not Secure:** If Eve has access to \( MAC_k \) or has old messages she knows \( k = 7 \).

**Eve can Forge:** If Eve has key \( k \) then she can forge.
Example of a MAC

1. \( k \in \{0, 1\}^n \) unif.
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Need: A function \( f \) such that knowing \( f \) on a few values does not reveal what \( f \) is.
We have them! Pseudo-Random Functions!
Construction of a Fixed Length MAC

Message are of length $n$
Let $F$ be a PRF from $\{0, 1\}^n$ to $\{0, 1\}^n$.

MAC:

1. $GEN$: choose a uniform key $k \in \{0, 1\}^n$ for $F$
2. $MAC_k(m)$: output $F_k(m)$
3. $V_k(m, t)$: output 1 iff $F_k(m) = t$

Theorem: $\Pi$ is a non-forgeable MAC
Proof Sketch: If forgeable then $F_k$ would not be pseudorandom.
Issue: We have not defined forgeable formally and we won’t.
Drawbacks?

- This only works for \textit{fixed-length} messages

- Since need tag $t$ to be short, this only works for \textit{short} messages

To get variable length we need a new Hardness Assumption.
Collision Resistant Hash Functions (CRHF)

Informal Def: A function $H$ from $\{0, 1\}^n$ to $X$ where $X$ is finite is Collision Resistant if it is HARD to find $x, y$ such that $H(x) = H(y)$.

Common Hardness Assumption: There exists Collision Resistant Hash Functions.

Often keyed: $H_k$ where $k$ is a key. $k$ of length $n$ gives $H$ on $\{0, 1\}^n$. 
Def: The Random Oracle Model is the Hardness Assumption that there exists a $H$ such that both:

- $H$ is a Collision Resistant Hash Functions.
- $H$ is a Pseudorandom function.

Often keyed: $H_k$ where $k$ is a key.
Random Oracle Model: Warning

Compare the following Hardness Assumptions:

- Factoring is hard. Well tested. Fermat (1600’s) worked on it! Easy to increase security parameter.
- RSA assumption. Worked on since 1978. But 40 years of modern math is a lot. Easy to increase security parameter.
- ROM. Hmmm. No candidate for the RO has been that well tested. The assumption $H$ is random harder to test then Factoring is hard. Not clear how much increasing the security parameter will help.

But! There are real functions (in two slides) that are really being used that seem to satisfy ROM.
Possible CRHF

Security Parameter $n$

$H_k(x, y)$: $k$ encodes $(p, q, g, h)$ where

- $p, q$ is an $n$-bit primes (who would have guessed! :-))
- $g$ has a larger period mod $N = pq$ (omit how we can find such a $g$).
- $h$ is some other random element of $\mathbb{Z}_N$.

$H : \mathbb{Z}_N \times \mathbb{Z}_N \rightarrow \mathbb{Z}_N$ is defined by

$$H(k, y) = g^k h^y \pmod{N}$$

Note: This is fixed length, but can use bigger and bigger security parameters so considered to be a function on $\{0, 1\}^n$. 
More CRHF

The following are really used! The definitions are ugly (like Trivium).

<table>
<thead>
<tr>
<th>Hash Sch</th>
<th>Year</th>
<th>Const</th>
<th>Numb bits</th>
<th>Year</th>
<th>Broken</th>
</tr>
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<tr>
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<td>1990</td>
<td>128</td>
<td></td>
<td>1995</td>
<td></td>
</tr>
<tr>
<td>MD5</td>
<td>1992</td>
<td>128</td>
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<td>1998</td>
<td></td>
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<tr>
<td>SHA1</td>
<td>1994</td>
<td>160</td>
<td></td>
<td>2005*</td>
<td></td>
</tr>
<tr>
<td>SHA-256</td>
<td>2005</td>
<td>256</td>
<td></td>
<td>Not Yet!</td>
<td></td>
</tr>
</tbody>
</table>

*SHA1 – collision found, but not quite broken.
Construction of a \( \geq n \)-length MAC

Message are of length \( \geq n \)
Let \( F_k \) be a PRF from \( \{0, 1\}^n \) to \( \{0, 1\}^n \).
Let \( H_k \) be a CRHF from \( \{0, 1\}^* \) to \( \{0, 1\}^n \).
Both keys are in \( \{0, 1\}^n \).
MAC:

1. \( \text{GEN} \): choose a uniform key \( k \in \{0, 1\}^n \) for \( F \) and \( H \)
2. \( \text{MAC}_k(m) \): output \( F_k(H_k(m)) \)
3. \( \text{V}_k(m, t) \): output 1 iff \( F_k(H_k(m)) = t \)

Theorem: \( \Pi \) is a non-forgable MAC
Proof Sketch: If forgeable then \( F_k \) would not be psuedorandom OR \( H_k \) would not be CRHF.
Issue: We have not defined forgeable formally and we won’t.
Drawbacks?

Alice: Bob, you signed a document saying you owe me $100,000

Bob: I didn’t! And even if I did you can’t prove it!
(Why do criminals on TV shows always say You can’t prove I’m guilty? They should just lie and say I didn’t do it.)

Need for the signature to be public!
Digital Signatures
Digital signatures

1. MAC uses private Key
2. MAC is good if Alice and Bob’s only enemy is Eve.
3. MAC is bad if Bob says I didn’t send that

Need a public key version of MAC that witnesses can verify.
Comparison to MACs?

- **Public verifiability**
  - “Anyone” can verify a signature
  - (Only a holder of the key can verify a MAC tag)

- **Transferable**
  - Can forward a signature to someone else . . .

- **Non-repudiation** Bob can’t deny he signed!
A signature scheme is defined by three PPT algorithms (GEN, SIGN, V):

- **GEN**: takes as input $1^n$, outputs $sk, pk \in \{0, 1\}^n$ (Secret Key, Public Key).

- **SIGN**: takes as input a private key $sk$ and a message $m \in \{0, 1\}^*$; outputs a signature $\sigma$

  $$\sigma \leftarrow \text{SIGN}_{sk}(m)$$

- **V**: takes a public key $pk$, message $m$, and signature $\sigma$ as input; outputs 1 or 0

  $$\forall m, pk, sk[V_{pk}(m, \text{SIGN}_{sk}(m)) = 1]$$
First Attempt at a Signature schemes

1. \( \text{GEN} \) generates primes \( p, q \) of length \( n \). \( p, q \) is private, \( N = pq \) is public. Let \( R = (p - 1)(q - 1) \). \( e, d \) such that \( ed \equiv 1 \pmod{R} \). \( e \) public, \( d \) private.

2. For Bob to sign message \( m \), Bob sends \( \sigma = m^d \pmod{N} \).

3. To verify Alice computes \( \sigma^d \)

\[
\sigma^e \equiv (m^d)^e \equiv m^{ed} \equiv m^{ed} \pmod{R} \equiv m \pmod{N}.
\]
Looks Secure But Its Not

There are attacks on it that work.

Omitted.

But what to do?

Just a small adjustment.
Second Attempt at a Signature schemes

Assume the Random Oracle Model. Assume Let $H$ be a Random Oracle.

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2. For Bob to sign message $m$, Bob sends $\sigma = H(m)^d \pmod{N}$.

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$$\sigma^e \equiv (H(m)^e)^d \equiv H(m)^{ed} \equiv H(m)^{ed} \pmod{R} \equiv H(m) \pmod{N}.$$ 

Secure?
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Theorem: If a message can be forged then $H$ is not a Random Oracle.
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Secure!