

The Vigenère Cipher, Matrix Cipher, Issues, and One-Time Pad

Lecture 03

The Vigenère cipher

Key: A word or phrase. Example: $dog = (3,14,6)$.

Easy to remember and transmit.

Example using *dog*.

Shift 1st letter by 3

Shift 2nd letter by 14

Shift 3rd letter by 6

Shift 4th letter by 3

Shift 5th letter by 14

Shift 6th letter by 6, etc.

Jacob Prinz is a Physics Major

Jacob Prinz isaPhysics Major

encrypts to

MOIRP VUWTC WYDDN BGOFG SDXUU

The Vigenère cipher

Key: $k = (k_1, k_2, \dots, k_n)$.

Encrypt (all arithmetic is mod 26)

$$\text{Enc}(m_1, m_2, \dots, m_N) =$$

$$m_1 + k_1, m_2 + k_2, \dots, m_n + k_n,$$

$$m_{n+1} + k_1, m_{n+2} + k_2, \dots, m_{n+n} + k_n,$$

...

Decrypt Decryption just reverse the process

The Vigenère cipher

- ▶ Size of key space?
 - ▶ If keys are 14-char then key space size $26^{14} \approx 2^{66}$
 - ▶ If variable length keys, even more.
 - ▶ Brute-force search infeasible
- ▶ Is the Vigenère cipher secure?
- ▶ Believed secure for many years. . .
- ▶ Might not have even been secure then. . .

Cracking Vig cipher: Step One-find Keylength

Assume T is a text encoded by Vig, key length L unknown.
For $0 \leq i \leq L - 1$, letters in pos $\equiv i \pmod{26}$ – same shift.
Look for a sequence of (say) 3-letters to appear (say) 4 times.

Example: aiq appears in the

57-58-59th slot, 87-88-89th slot 102-103-104th slot
162-163-164th slot

Important: Very likely that aiq encrypted the same 3-letter sequence and hence the length of the key is a divisor of

$87-57=30$ $102-87=15$ $162-102=60$

The only possible L 's are 1,3,5,15.

Good Enough: We got the key length down to a small finite set.

Important Point about letter Freq

Assume (and its roughly true): In an English text of length N :

e occurs $\sim 13\%$ t occurs $\sim 9\%$ a occurs $\sim 8\%$

Etc- other letters have frequencies that are true for all texts.

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Relevant to us:

\vec{q} freq of every L th letter: then $\sum_{i=1}^{26} q_i^2 \approx 0.065$.

\vec{q} is NOT (we won't define that rigorously): $\sum_{i=1}^{26} q_i^2$ MUCH lower.

Cracking Vig cipher: Step One-find Keylength

Let K be the set of possible key lengths. K is small. For every $L \in K$:

- ▶ Form a stream of every L th character.
- ▶ Find the frequencies of that stream: \vec{q} .
- ▶ Compute $Q = \sum q_i^2$
- ▶ If $Q \approx 0.065$ then YES L is key length.
- ▶ If Q much less than 0.065 then NO L is not key length.
- ▶ One of these two will happen
- ▶ Just to make sure, check another stream.

Note: Differs from [Is English](#):

[Is English](#) wanted to know if the text was actually English
What we do above is see if the text has same dist of English, but okay if diff letters. E.g., if z is 13%, a is 9%, and other letters have roughly same numbers as English then we know the stream is SOME Shift. We later use [Is English](#) to see which shift.

A Note on Finding Keylength

We presented Method ONE:

1. Find phrase of length x appearing y times. Differences D .
2. K is set of divisors of all $L \in D$. Correct keylength in K .
3. Test $L \in K$ for key length until find one that works.

Or could try all key lengths up to a certain length, Method TWO:

1. Let $K = \{1, \dots, 100\}$ (I am assuming key length ≤ 100).
2. Test $L \in K$ for key length until find one that works.

Note: With modern computers use Method TWO. In days of old eyeballing it made Method ONE reasonable.

Cracking the Vig cipher: Step Two-Freq Anal

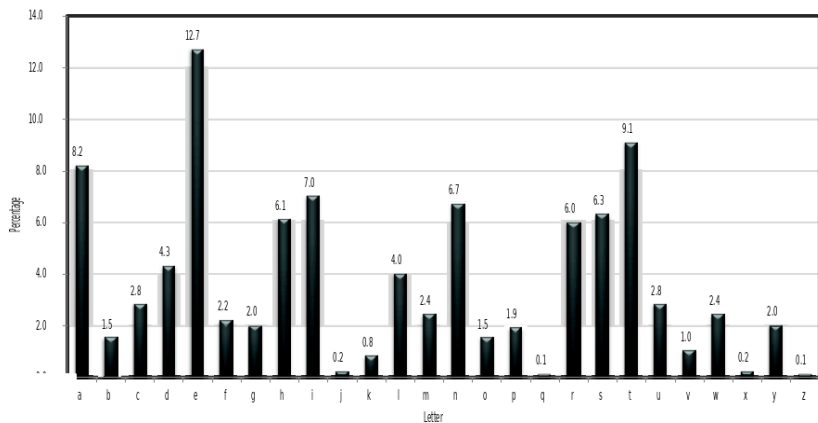
After Step One we have the key length L . Note:

- ▶ Every L^{th} character is “encrypted” using the same shift.
- ▶ **Important:** Letter Freq still hold if you look at every L 14th letter!

Step Two:

1. Separate text T into L streams depending on position mod L
2. For each steam try every shift and use **Is English** to determine which shift is correct.
3. You now know all shifts for all positions. Decrypt!

Using plaintext letter frequencies



Byte-wise Vigenère cipher

- ▶ The key is a string of bytes
- ▶ The plaintext is a string of bytes
- ▶ To encrypt, XOR each character in the plaintext with the next character of the key
 - ▶ Wrap around in the key as needed
- ▶ Decryption just reverses the process.

Note: Decryption and Encryption both use XOR with same key.

Note: Can be cracked as original Vig can be cracked.

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 $3x = 201$, so $x = 67$.

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Note: $-12, \frac{1}{3}$ are intermediaries. Want result in $\{0, \dots, n - 1\}$.

Gen 2-letter Sub and Matrix Codes

Lecture 03

Shift, Affine, Vig, Gen Sub, Easy to Crack

Shift, Affine, Vig all 1-letter substitutions. Freq cracked them.

Idea: Lets substitute two letters at a time.

An Idea Which History Passed By:

Definition: Gen Sub 2-Cipher with perm f on $\{0, \dots, 25\}^2$.

1. Encrypt via $xy \rightarrow f(xy)$.
2. Decrypt via $xy \rightarrow f^{-1}(xy)$

Why never used?

1. It was used but they kept it hidden and still not known!
2. The key length is roughly $26^2 \times 10 = 6760$ bits.
3. Old days: hard to use. Now: easy to crack.

Need bijection of $\{0, \dots, 25\} \times \{0, \dots, 25\}$ that is easy to use.

The Matrix Cipher

Definition: Matrix Cipher. Pick M a 2×2 matrix.

1. Encrypt via $xy \rightarrow M(xy)$.
2. Decrypt via $xy \rightarrow M^{-1}(xy)$

Encode: Break T into blocks of 2, apply M to each pair.

Decode: Do the same only with M^{-1} .

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Is Bill punking you ... again? No he is not.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$M^{-1} = \frac{1}{ad - bc} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Do you recognize the expression $ad - bc$?

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Do you recognize the expression $ad - bc$? Determinant!

Inverse Matrix in \mathbb{C} and in Mods

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1. Matrix M over \mathbb{C} has an inverse iff $ad - bc \neq 0$.
2. Matrix M over Mod n has an inverse iff $ad - bc$ is rel prime to n iff $ad - bc$ has an inverse in Mod n .
3. Matrix M over Mod 26 has an inverse iff $ad - bc$ is rel prime to 26 iff $ad - bc$ has no factors of 2 or 13 iff has an inverse in Mod 26.

Stuff to know for Special Lecture on Sept 24:

1. A matrix is invertible iff all of the rows are linearly ind.
2. If over \mathbb{Z}_p where p is a prime then more like \mathbb{C} - all numbers have inverses so need $ad - bc \neq 0$.

The Matrix Cipher

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Good News:

1. Can test if M^{-1} exists, and is so find it, easily.
2. M small, so Key small.
3. Applying M or M^{-1} to a vector is easy computationally.

Bad News:

1. Eve CAN crack using frequencies of pairs of letters.
2. Eve CAN crack – Key space has $< 26^4 = 456976$. Small.

So what to do?

The Matrix Cipher

Definition: Matrix Cipher. Pick n and M an $n \times n$ matrix with \det rel prime to 26.

1. Encrypt via $\vec{x} \rightarrow M(\vec{x})$.
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We'll take $n = 30$.

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1. Can determine if M has inv and if so find it easily.
2. M still small, so Key small.
3. Applying M or M^{-1} to a vector is easy computationally.
4. Eve can crack using freqs of 30-long sets of letters? Hard?
5. Eve cannot use brute force – Key Space is $\sim 26^{900}$.

Is Matrix Cipher Uncrackable?

VOTE: Yes, No, Unknown to Science, Other.

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So why is it not used? Discuss!

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Seems to be **Unknown to Science!**
So why is it not used? Discuss!
3. In reality Eve has prior messages and what they coded to, so from that she can easily crack it. (Next Slide.) **That is why not used.**

Cracking Matrix Cipher

Example using 2×2 Matrix Cipher.

Eve learns that $(19,8)$ encrypts to $(3,9)$. Hence:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 19 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

So

$$19a + 8b = 3$$

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Two linear equations, Four variables

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If Eve learns one more 2-letter message decoding then she will have

Four linear equations, Four variables

which she can solve! Yeah? Boo? Depends whose side you are on.

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Let C be any of Shift, Affine, Vigenere, Matrix.

Recall: C is crackable if text is long enough.

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Even knows that the city and state are the same!

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Alabama*, Arizona*, Arkansas, California, Colorado*, Delaware, Florida, New Georgia*, Idaho, Illinois*, Indianapolis, Iowa, Jersey, Kansas, Maryland*, Minneapolis, Minnesota, Mississippi*, Missouri, Montana, Nebraska, Nevada*, New York, Ohio, Oklahoma, Oregon, Tennessee*, Texas, Utah*, Virginia*, Virginia Beach, Wisconsin Dells, Wisconsin Rapids.

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There are 33 such cities, 22 of which still exist.
Eve's search for the spy is reduced!

How to Fix This?

Problem: If C is any of the ciphers discussed then Eve can tell when two messages are the same.

Discuss: Is there a cipher for which Eve cannot tell this?

See Next Lecture!