MIDTERM REVIEW-ADMIN

Midterm Review-Admin

1) Midterm is Monday Oct 29 2-3:15 in class
2) Can bring one sheet of notes.
   Can use both sides
   Can be typed
   You can put whatever you want on it.
   Can copy a classmates and use it but thats stupid
   Can try to cram the entire course onto it but thats stupid
3) No calculators allowed. Numbers will be small.
4) Coverage: Slides/HW.
5) Not on Exam: Guest lectures.
6) We hope to grade it and post it Monday Night.
7) If can't take the exam tell me ASAP.
8) Advice: Understand rather than memorize.

MIDTERM REVIEW-CONTENT

Alice, Bob, and Eve

- Alice sends a message to Bob in code.
- Eve overhears it.
- We want Eve to not get any information.

There are many aspects to this:

- Information-Theoretic Security.
- Computational-Theoretic Security (Hardness Assumption)
- The NY,NY problem: Do not always code m the same way.
- Private Key or Public key
- Kerckhoff's principle: Eve knows cryptosystem.
- History: How much computing power does Eve have?

The First Steps in Any Cipher

1. Get rid of spacing
2. Get rid of punctuation
3. Make everything capitol letters
4. Class convention: we use either
   4.1 $\Sigma = \{A, \ldots, Z\}$, or
   4.2 $\Sigma = \{0, \ldots, n-1\}$ (n is often p a prime), or
   4.3 $\Sigma = \{0, 1\}$.

Private Key Ciphers

Single Letter Sub Ciphers

1. Shift cipher: $f(x) = x + s$. $s \in \{0, \ldots, 25\}$.
2. Affine cipher: $f(x) = ax + b$. $a, b \in \{0, \ldots, 25\}$. a rel prime 26.
3. Keyword Shift: From keyword and shift create random-looking perm of $\{a, \ldots, z\}$.
4. Keyword Mixed: From keyword create random-looking perm of $\{a, \ldots, z\}$.
5. Gen Sub Cipher: Take random perm of $\{a, \ldots, z\}$.

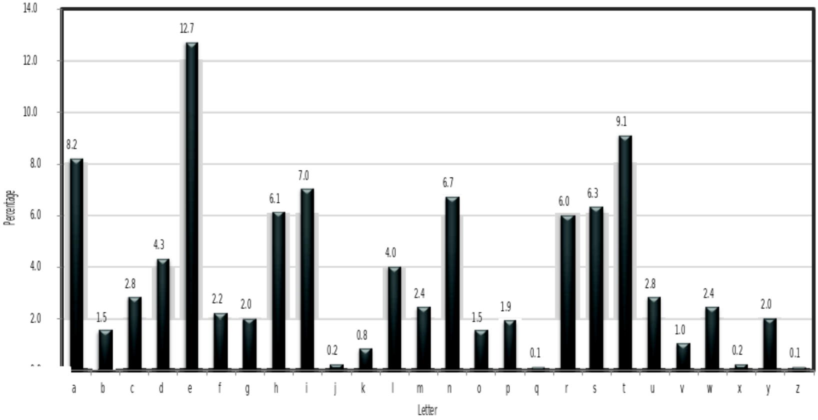
All Single Letter Sub Ciphers Crackable

Important: Algorithm Is-English.

1. Input(T) a text
2. Find f_T, the freq vector of T
3. Find $x = f_T \cdot f_E$ where f_E is freq vector for English
4. If $x \geq 0.06$ then output YES. If $x \leq 0.04$ then output NO. If $0.04 < x < 0.06$ then something is wrong.

1. Shift , Affine have small key space: can try all keys and see when Is-English says YES.
2. For others use Freq analysis, e.g., e is most common letter.
3. If message is numbers (e.g., Credit Cards) or ASCII (e.g., Byte-Shift) there are still patterns so can use freq analysis.

Letter Frequencies



Randomized Shift

How to NOT encode the same m the same way:

Randomized shift: Key is a function $f : S \rightarrow S$.

1. To send message $(m_1, \ldots, m_L)$ (each m_i is a character)
   1.1 Pick random $r_1, \ldots, r_L \in S$. For $1 \le i \le L$ compute $s_i = f(r_i)$.
   1.2 Send $((r_1; m_1 + s_1), \ldots, (r_L; m_L + s_L))$
2. To decode message $((r_1; c_1), \ldots, (r_L; c_L))$
   2.1 For $1 \le i \le L$ $s_i = f(r_i)$.
   2.2 Find $(c_1 - s_1, \ldots, c_L - s_L)$

   Note: Can be cracked.

Example

The key is $f(r) = 2r + 7$. Alice wants to send NY,NY which we interpret as nyny.
Need four shifts.

Pick random $r = 4$, so first shift is $2 * 4 + 7 = 15$
Pick random $r = 10$, so second shift is $2 * 10 + 7 = 1$
Pick random $r = 1$, so third shift is $2 * 1 + 7 = 9$
Pick random $r = 17$, so fourth shift is $2 * 17 + 7 = 15$

Send (4;C), (10,Z), (1,W), (17,N)

Eve will not be able to tell that is of the form XYXY.

The Vigenère cipher

Key: $k = (k_1, k_2, \ldots, k_n)$.
Encrypt (all arithmetic is mod 26)

$$Enc(m_1, m_2, \ldots, m_N) =$$

$$m_1 + k_1, m_2 + k_2, \ldots, m_n + k_n,$$

$$m_{n+1} + k_1, m_{n+2} + k_2, \ldots, m_{n+n} + k_n,$$

$$\ldots$$

Decrypt Decryption just reverse the process

Cracking Vig cipher

1. Find Keylength or set K of them. Either try length 1,2,3,... or find repeated strings of letters so can guess.
2. Let K be the set of possible key lengths. For every $L \in K$:
   - 2.1 Separate text T into L streams depending on position mod L
   - 2.2 For each steam try every shift and use Is-English to determine which shift is correct.
   - 2.3 You now know all shifts for all positions. Decrypt!

Getting More Out of Your Phrase

If the key was Corn Flake key of length 9. Want More.
We form a key of length $LCM(4,5) = 20$.

C	O	R	N	C	O	R	N	C	O	R	N	C	O	R	N	C
F	L	A	K	E	F	L	A	K	E	F	L	A	K	E	F	L
7	25	17	23	6	19	2	13	12	18	22	24	2	24	21	18	1

ADD it up to get new 20-long key.

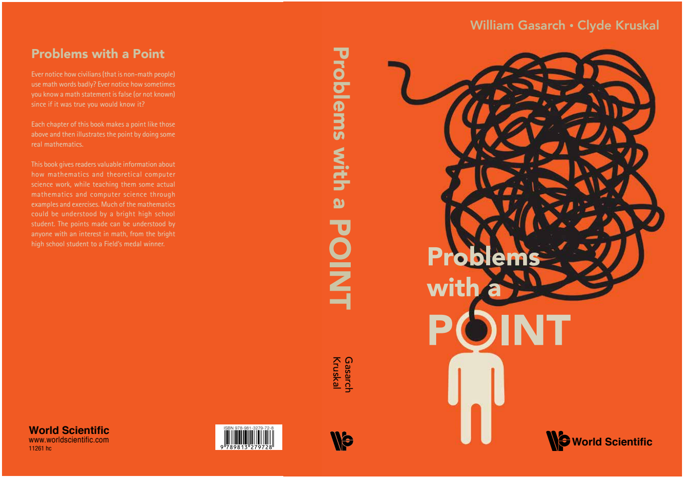
Wheel of Fortune yields key size $LCM(5,2,7) = 70$.
Crackable? in 2018 YES, in 1776 Probably Not.

Vig Book Cipher

Use Book for key.

1. LONG key- great!
2. Should pick obscure book (see next slide).
3. Crackable NOW by looking at common pairs-of-letters since both book and message are English.
4. Probably hard in 1776.

An Obscure Book You Can Use



Shift, Affine, ... Easy to Crack

1. Shift
2. Affine
3. Keyword Shift
4. Keyword Mixed
5. Gen Sub
6. Vig
7. all 1-letter substitutions.

Freq cracked them (for Vig Freq plus some other stuff).

Idea: Sub n letters at a time.

Need bijection of $\{0, \ldots, 25\}^n$ to $\{0, \ldots, 25\}^n$ that is easy to use.

The Matrix Cipher

Definition: Matrix Cipher. Pick n and M an $n \times n$ invertible matrix.

1. Encrypt via $\vec{x} \to M(\vec{x})$.
2. Decrypt via $\vec{y} \to M^{-1}(\vec{y})$

We'll take $n = 30$.

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1. Easy for Alice and Bob.
2. Key M is small enough to be easy for Alice and Bob but too large for Eve to use brute force.
3. Eve can crack using freqs of 30-long sets of letters? Hard?
4. Ciphertext only might be uncrackable.
5. Can crack from message-cipher pairs.

The Matrix Cipher: Ciphertext Only

If $n \times n$ matrix then keyspace has roughly 26^{n^2}.

1. Trying every matrix takes 26^{n^2}.
2. If guess one row at a time then $O(n26^n)$.
3. Lesson: Eve may think of attacks you had not thought of.
4. Lesson: Attacks can be thwarted, once known, by increasing n

One-time pad

1. Let $\mathcal{M} = \{0,1\}^n$

2. *Gen*: choose a uniform key $k \in \{0,1\}^n$

3. $Enc_k(m) = k \oplus m$

4. $Dec_k(c) = k \oplus c$

5. Correctness:

$$
\begin{aligned}
Dec_k(Enc_k(m)) &= k \oplus (k \oplus m) \\
&= (k \oplus k) \oplus m \\
&= m
\end{aligned}
$$

PROS AND CONS Of One-time pad

1. If Key is N bits long can only send N bits.
2. $\oplus$ is FAST!
3. The one-time pad is uncrackable. YEAH!
4. Generating truly random bits is hard. BOO!
5. Psuedo-random can be insecure – I did example.

Public Key Ciphers
Eve can go . . .

Public Key Cryptography

Alice and Bob never have to meet!

NT Algorithms needed for Public Key

All arithmetic is mod p. The following can be done quickly.

1. Given (a, n, p) compute $a^n \pmod{p}$. Repeated Squaring. (1) $\leq 2 \lg n$ always, (2) $\leq \lg n + O(1)$ if n close to 2^{2^m}.
2. Given n, find a safe prime of length n and a generator g.
3. Given a, b rel prime find inverse of a mod b: Euclidean alg.
4. Given $a_1, \ldots, a_L$ and $b_1, \ldots, b_L$, b_i's rel prime, find $x \equiv a_i \pmod{b_i}$.
5. Given (a, p) find $\sqrt{a}$'s. We did $p \equiv 3 \pmod{4}$ case.
6. Given (a, N) and p, q such that $N = pq$, find $\sqrt{a}$'s.

Number Theory Assumptions

1. Discrete Log is hard.
2. Factoring is hard.
3. Given (a, N), find $\sqrt{a}$ without being given factors of N is hard. (This is equiv to factoring.)

Note: We usually don't assume these but instead assume close cousins.

The Diffie-Helman Key Exchange

Alice and Bob will share a secret s.

1. Alice finds a (p, g), p of length n, g gen for $\mathbb{Z}_p$. Arith mod p.
2. Alice sends (p, g) to Bob in the clear (Eve can see it).
3. Alice picks random $a \in \{1, \ldots, p-1\}$. Alice computes g^a and sends it to Bob in the clear (Eve can see it).
4. Bob picks random $b \in \{1, \ldots, p-1\}$. Bob computes g^b and sends it to Alice in the clear (Eve can see it).
5. Alice computes $(g^b)^a = g^{ab}$.
6. Bob computes $(g^a)^b = g^{ab}$.
7. g^{ab} is the shared secret.

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Definition

Let f be $f(p, g, g^a, g^b) = g^{ab}$.

Hardness assumption: f is hard to compute.

How Useful is Diffie-Helman

CAVEAT: DH is not a cipher.

PRO: Alice and Bob can use g^{ab} to transmit a key for a cipher.

Used? DH is at used in many real authentication schemes!

ElGamal is DH with a Twist

1. Alice and Bob do Diffie Helman.
2. Alice and Bob share secret $s = g^{ab}$.
3. Alice and Bob compute $(g^{ab})^{-1} \pmod{p}$.
4. To send m, Alice sends $c = mg^{ab}$
5. To decrypt, Bob computes $c(g^{ab})^{-1} \equiv mg^{ab}(g^{ab})^{-1} \equiv m$

We omit discussion of Hardness assumption (HW)

Needed Math for RSA – The ϕ Function

Definition

$\phi(n)$ is the numb of nums in $\{1, \ldots, n-1\}$ that are rel prime to n.

Note: If p is prime then $\phi(p) = p - 1$.

Known: If n is any number then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Ramifications: For all m, $a^m \equiv a^{m \pmod{\phi(n)}} \pmod{n}$.

Known: If a, b are relatively prime then $\phi(ab) = \phi(a)\phi(b)$.

Known: Given R, easy to find e rel prime to R and d such that $ed \equiv 1 \pmod{R}$.

Believe: Let $N = pq$, $R = (p-1)(q-1)$ and e rel prime to R. If know N but Not R then hard to find d with $ed \equiv 1 \pmod{R}$.

RSA

Let n be a security parameter

1. Alice picks two primes p, q of length n and computes $N = pq$.
2. Alice computes $\phi(N) = \phi(pq) = (p-1)(q-1)$. Denote by R
3. Alice picks an $e \in \{\frac{R}{3}, \ldots, \frac{2R}{3}\}$ that is relatively prime to R. Alice finds d such that $ed \equiv 1 \pmod{R}$.
4. Alice broadcasts (N, e). (Bob and Eve both see it.)
5. Bob: To send $m \in \{1, \ldots, N-1\}$, send $m^e \pmod{N}$.
6. If Alice gets $m^e \pmod{N}$ she computes

$$(m^e)^d \equiv m^{ed} \equiv m^{ed \pmod{R}} \equiv m^{1 \pmod{R}} \equiv m$$

Hardness Assumption for RSA

Definition: Let f be $f(N, e) = d$, where $N = pq$, and

$$ed \equiv 1 \pmod{(p-1)(q-1)}$$

Hardness assumption (HA): f is hard to compute.

Plain RSA Bytes!

The RSA given above is referred to as Plain RSA.
Insecure! m is always coded as $m^e \pmod{N}$.

Make secure by padding: $m \in \{0,1\}^{L_1}$, $r \in \{0,1\}^{L_2}$.

To send $m \in \{0,1\}^{L_1}$, pick rand $r \in \{0,1\}^{L_2}$, send $(rm)^e$.
(NOTE- rm means r CONCAT with m here and elsewhere.)
DEC: Alice finds rm and takes leftmost L_1 bits.
Caveat: RSA still has issues when used in real world. They have been fixed. Maybe.

Attacks on RSA

1. Factoring Algorithms. We saw some ideas with Jevon's Number. Response: Pick larger p, q
2. If Zelda give A_i (N_i, e):
   2.1 Low-e attack: Response: High e. Duh.
   2.2 $m^e < N_1 \cdots N_L$: Response: Pad m.
3. If Zelda give A_i (N, e_i) and two of the e_i's are rel prime, then Euclidean Alg Attack: Response: Give everyone diff N's. Duh.
4. Timing Attacks: Response: Pad the amount of time used.

Caveat: Theory says use different e's. Practice says use $e = 2^{16} + 1$ for speed.

Math for Rabin Encryption – Procedures

How to find square roots mod p if $p \equiv 3 \pmod 4$.
All arithmetic is mod p.

Input(c)

Compute $c^{(p-1)/2}$. If it is NOT 1 then output There is no square root!. If it is 1 then goto next step

Compute $a = c^{(p+1)/4}$.

Output a and $p - a$. These are the two square roots.

Note: There is a similar algorithm for $p \equiv 1 \pmod 4$ but it is slightly more complicated.

Rabin's Encryption Scheme

n is a security parameter

1. Alice gen p, q primes of length n. Let $N = pq$. Send N.
2. Encode: To send m, Bob sends $c = m^2 \pmod{N}$.
3. Decode: Alice can find m such that $m^2 \equiv c \pmod{N}$.

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BIG PRO: Factoring Hard is hardness assumption.
CON: Alice has to figure out which of the sqrts is correct message.

A Theorem from Number Theory

Definition: A *Blum Int* is product of two primes $\equiv 3 \pmod 4$.
Example: $21 = 3 \times 7$.

Notation: SQ_N is the set of squares mod N. (Often called QR_N.)
Example: If $N = 21$ then $SQ_N = \{1, 4, 7, 9, 15, 16, 18\}$.

Theorem: Assume N is a Blum Integer. Let $m \in SQ_N$. Then of the two or four sqrts of m, only one is itself in SQ_N.
Proof: Omitted. Note: (1) not that hard, and (2) in Katz book.

We use Theorem to modify Rabin Encryption.

Rabin's Encryption Scheme 2.0

Also called The Blum-Williams Variant of Rabin
n is a security parameter.

1. Alice gen p, q primes of length n such that $p, q \equiv 3 \pmod 4$. Let $N = pq$. Send N.
2. Encode: To send m, Bob sends $c = m^2 \pmod N$. Only send m's in SQ_N.
3. Decode: Alice can find 2 or 4 m such that $m^2 \equiv c \pmod N$. Take the $m \in SQ_N$.

CON: Messages have to be in SQ_N.

History: Had timing been different Rabin Enc would be used.

Goldwasser-Micali Encryption

n is a security parameter. Will only send ONE bit. Bummer!

1. Alice gen p, q primes of length n, and $z \in NSQ_N$. Computes $N = pq$. Send (N, z).
2. Encode: To send $m \in \{0, 1\}$, Bob picks random $x \in \mathbb{Z}_N$, sends $c = z^m x^2 \pmod{N}$. Note that:
   2.1 If $m = 0$ then $z^m x^2 = x^2 \in SQ_N$.
   2.2 If $m = 1$ then $z^m x^2 = zx^2 \in NSQ_N$.
3. Decode: Alice determines if $c \in SQ$ or not. If YES then $m = 0$. If NO then $m = 1$.

BIG PRO: Hardness assumption natural: SQ_N hard.
BIG CON: Messages have to be 1-bit long.
TIME: For one bit you need $4 \log N$ steps.

Blum-Goldwasser Enc. n Sec Param, L length of msg

1. Alice: p, q primes len n, $p, q \equiv 3 \pmod 4$. $N = pq$. Send N.
2. Encode: Bob sends $m \in \{0,1\}^L$: picks random $r \in \mathbb{Z}_N$
   $x_1 = r^2 \mod N \qquad b_1 = LSB(x_1)$.
   $x_2 = x_1^2 \mod N \qquad b_2 = LSB(x_2)$.
   $\vdots$
   $x_L = x_{L-1}^2 \mod N \qquad b_L = LSB(x_L)$.
   Send $c = ((m_1 \oplus b_1, \ldots, m_L \oplus b_L), x_L)$.
3. Decode: Alice: From x_L Alice can compute x_{L-1}, ..., x_1 by sqrt (can do since Alice has p, q). Then can compute $b_1, \ldots, b_L$ and hence $m_1, \ldots, m_L$.

BIG PRO: Hardness assumption is BBS psuedo-random.
TIME: For L bits need $(L+3)\log N$ steps. Better than Goldwasser-Micali.

LWE-KE

1. LWE-KE is a protocol for Key Exchange that does not rely on Number Theory Hardness Assumption
2. There is also a LWE-RSA.
3. These might be useful if Factoring and Discrete Log can be done by a quantum computer.
4. My presentation of it was not quite right.
5. The literature on these is not quite right either.

Good Luck on the Exam

Good Luck on the Exam!