Problems with a Point: Exploring Math and Computer Science

William Gasarch
Clyde Kruskal
In 2003 Lance Fortnow started Complexity Blog
In 2007 Bill Gasarch joined and it was a co-blog.
In 2015 various book publishers asked us
Can you make a book out of your blog?
Lance declined but Bill said YES.
Book’s Point

Bill took the posts that had the following format:

▶ make a point about mathematics
▶ do some math to underscore those points

and made those into chapters.

Caveat: Not every chapter is quite like that. To quote Ralph Waldo Emerson: ‘A foolish consistency is the hobgoblin of small minds.’
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The publisher wisely decided to be less cute and more informative:
Problems with a Point: Exploring Math and Computer Science
Clyde Joins the Project!

After some samples of Bill’s writing the publisher said
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After some samples of Bill’s writing the publisher said

Please Procure People to Polish Prose and Proofs of Problems with a Point

so
Clyde Joins the Project!

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Clyde Kruskal became a co-author.
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Now onto some samples of the book!
From the Year 2000 Maryland Math Competition:

There are 2000 cans of paint. Show that at least one of the following two statements is true:

- There are at least 45 cans of the same color.
- There are at least 45 cans that are different colors.

Work on it.
From the Year 2000 Maryland Math Competition:

*There are 2000 cans of paint. Show that at least one of the following two statements is true:*

- There are at least 45 cans of the same color.
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Work on it.

**Answer:**

If there are 45 different colors of paint then we are done. Assume there are \( \leq 44 \) different colors. If all colors appear \( \leq 44 \) times then there are \( 44 \times 44 = 1936 < 2000 \) cans of paint, a contradiction.

**Note:** this was Problem 1, which is supposed to be easy and indeed 95% got it right. What about the other 5%? Next slide.
One of the Wrong Answers. Or is it?

There are 2000 cans of paint. Show that at least one of the following two statements is true:

- There are at least 45 cans of the same color.
- There are at least 45 cans that are different colors.

Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.

What do you think:

- Thats just stupid. 0 points.
- Question says cans of the same color... The full 30 pts.
- Not only does he get 30 points, but everyone else should get 0.
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▶ There are at least 45 cans of the same color.
▶ There are at least 45 cans that are different colors.

If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.

What do you think:
▶ That's just stupid. 0 points.
▶ Well...
  he's got a point. 30 points in fact.
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From the year 2007 Maryland Math Competition.

QUESTION: Let $ABC$ be a fixed triangle. Let $COL$ be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle $DEF$ in the plane such that $DEF$ is similar to $ABC$ and the vertices of $DEF$ all have the same color.
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**Note** I think I was assigned to grade it since it looks like the kind of problem I would make up, even though I didn’t. It was problem 5 (out of 5) and was hard. About 100 students tried it, 8 got full credit, 10 got partial credit.
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Funny Answer One:
All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like $2 + 2 = 5$ if thats what my math teacher says. Math is pretty subjective anyway.
Was Student One Serious?

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**Theorem** The students is not serious.

**Proof** Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don’t really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.
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Was Student Two Serious. Yes. about Justice!
The Real Answer to Points in the Plane Problem

Each point in the plane is colored either red or green. Let $ABC$ be a fixed triangle. Prove that there is a triangle $DEF$ in the plane such that $DEF$ is similar to $ABC$ and the vertices of $DEF$ all have the same color.

Fix a 2-coloring of the plane.
There are 3 equally-spaced mono points on \( x \)-axis

**Proof** Clearly there are two points on the \( x \)-axis of the same color: \( p_1, p_2 \) are RED. If \( p_3 \), the midpoint of \( p_1, p_2 \), is RED then \( p_1, p_3, p_2 \) are all RED. DONE. Hence we assume \( p_3 \) is GREEN.

Let \( p_4 \) be such that \( |p_1 - p_4| = |p_2 - p_1| \). If \( p_4 \) is RED then \( p_4, p_1, p_2 \) are all RED. DONE. Hence we assume \( p_4 \) is GREEN.

Let \( p_5 \) be such that \( |p_5 - p_2| = |p_2 - p_1| \). If \( p_5 \) is RED then \( p_1, p_2, p_5 \) are all RED. DONE. Hence we assume \( p_5 \) is GREEN.

Only case left \( p_3, p_4, p_5 \) are all GREEN. DONE.
Figure: Triangle Similar to $ABC$ with Monochromatic Vertices

$P, Q, R$ are RED.

If $T$ or $U$ or $S$ are RED then get RED Triangle similar to $ABC$.

If not then ALL of $T, U, S$ are GREEN, so get GREEN triangle similar to $ABC$. 
Point: What is a Simple Function?

Bill assigned the following in Discrete Math: For each of the following sequences find a simple function $A(n)$ such that the sequence is $A(1), A(2), A(3), \ldots$

1. 10, -17, 24, -31, 38, -45, 52, \ldots
2. -1, 1, 5, 13, 29, 61, 125, \ldots
3. 6, 9, 14, 21, 30, 41, 54, \ldots

Caveat: These are NOT trick questions. Work on it.
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1. 10, -17, 24, -31, 38, -45, 52, ··· $A(n) = (-1)^{n+1}(7n + 3)$.

2. -1, 1, 5, 13, 29, 61, 125, ··· $A(n) = 2^n - 3$.

3. 6, 9, 14, 21, 30, 41, 54, ··· $A(n) = n^2 + 5$. 
A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

*We never defined **Simple Function** in class so I went to Wikipedia. It said that **A Simple Function is a linear combination of indicator functions of measurable sets**. *Is that what you want us to use?*
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I doubt the student knows what those terms mean.
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The student got the first one right, but left the other two blank.
The last question brings up the question of when patterns do and don’t hold. We looked for cases where a pattern did not hold.
First Non-Pattern: \( n \) Points on a circle

What is the max number of regions formed by connecting every pair of \( n \) points on a circle. For \( n = 1, 2, 3, 4, 5 \):

Tempted to guess \( 2^{n-1} \).
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But for \( n = 6 \), the number of regions is only 31.
First Non-Pattern: $n$ Points on a circle

What is the max number of regions formed by connecting every pair of $n$ points on a circle. For $n = 1, 2, 3, 4, 5$:

Tempted to guess $2^{n-1}$.
But for $n = 6$, the number of regions is only 31.
The actual number of regions for $n$ points is $\binom{n}{4} + \binom{n}{2} + 1$. 
Second Non-Pattern: Borwein Integrals

\[ \int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2} \]

\[ \int_0^\infty \frac{\sin x \sin \frac{x}{3}}{x} = \frac{\pi}{2} \]

\[ \int_0^\infty \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13}}{x \frac{x}{3} \frac{x}{5} \frac{x}{7} \frac{x}{9} \frac{x}{11} \frac{x}{13}} = \frac{\pi}{2} \]

But

\[ \int_0^\infty \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13}}{x \frac{x}{3} \frac{x}{5} \frac{x}{7} \frac{x}{9} \frac{x}{11} \frac{x}{13}} \approx 0.9999999999852937186 \times \frac{\pi}{2} \]
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\]

\[
\frac{4678079247134407386537864469\pi}{935615849440640907310521750000} \sim 0.99999999999852937186 \times \frac{\pi}{2}
\]
Why the breakdown at 15?

Because

\[ \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1 \]

but

\[ \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} > 1. \]

For more Google

Borwein Integral
Bill found a pamphlet:

**The Julia Robinson Mathematics Festival:**
**A Sample of Mathematical Puzzles**
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

*How can you divide and distribute 5 muffins to 3 students so that every student gets \( \frac{5}{3} \) where nobody gets a tiny sliver?*
## Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
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<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
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<tr>
<td>Carol</td>
<td>GREEN</td>
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**Smallest Piece:** $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Work no it
### Five Muffins, Three People—Proc by Picture

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

NO WE CAN’T!
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

(Henceforth: All muffins are cut into $\geq 2$ pieces.)

Case 1: Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.

(Henceforth: All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
The Muffin Problem:

How can you divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

This Problem went from recreational Mathematics to Serious Math when we replaced $(5,3)$ with $(m,s)$. 
Is the Muffin Problem Interesting?

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2. Book with Four Authors (To appear in 2020)
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