BILL, RECORD LECTURE!!!!

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Gen 2-letter Sub and Matrix Codes

September 28, 2020

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2. The key length is roughly $26^2 \times 10 = 6760$ bits.
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Need bijection of $\{0, \ldots, 25\} \times \{0, \ldots, 25\}$ that is easy to use.

The Matrix Cipher

Def Matrix Cipher. Pick M a 2×2 matrix.

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Do you recognize the expression $ad - bc$? Determinant!

Inverse Matrix in $\mathbb{C}$ and in Mods

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1. Matrix M over $\mathbb{C}$ has an inverse iff $ad - bc \neq 0$.
2. Matrix M over Mod n has an inverse iff $ad - bc$ is rel prime to n iff $ad - bc$ has an inverse in Mod n.
3. Matrix M over Mod 26 has an inverse iff $ad - bc$ is rel prime to 26 iff $ad - bc$ has no factors of 2 or 13 iff has an inverse in Mod 26.

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So what to do?

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Def Pick $n \in \mathbb{N}$ and M an $\mathbf{n} \times \mathbf{n}$ matrix with det rel prime to 26.

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1. M still small, so Key small.
2. Finding M^{-1}, mult by M or M^{-1} fast.
3. Eve cannot use brute force. Key Space is $\sim 26^{64} \sim 10^{90}$, Number of protons is $\sim 10^{79}$. (the number of non-invertible matrices is very small so 26^{64} is a good approximation).

Lets Try Brute Force Even if Slow

1. Input T, a coded text.
2. For EVERY 8×8 invertible matrix M over $\mathbb{Z}_{26}$,
   - 2.1 Decode T into T' using M.
   - 2.2 IF LOOKS-LIKE-ENGLISH(T')=YES then STOP and output T', else goto next matrix M.

Takes roughly 26^{64} steps.

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YES- we can do 8×26^8.

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Let $Mt_i = m_i$. Then $(1, 1, 7) \cdot t_i = m_i^1$ is first letter of m_i.

$$(m_1^1, m_2^1, m_3^1, \ldots, m_N^1)$$

is every third letter. Can do IS-ENGLISH on it.

Can Crack in 8×26^8

Eve knows that Alice and Bob decode with 8×8 Matrix M.
Ciphertext is

$$T = t_1 t_2 \cdots t_N \qquad t_i = t_i^1 \cdots t_i^8$$

For $i = 1$ to 8

For all $r \in \mathbb{Z}_{26}^8$ (guess that r is ith row of B).

$T' = (r \cdot t_1, \ldots, r \cdot t_N)$ (Is every 8th letter.)

IF IS-ENGLISH(T')=YES then $r_i = r$ and goto next i. Else goto the next r.

M is

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ r_1 & \cdots & r_n \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

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The row-by-row method takes $O(n26^n)$.

Important Lesson

Assume: 26^{64} time is big enough to thwart Eve.

1. If we think that best Eve can do is $O(26^{n^2})$ then we take $n = 8$, so Eve needs $O(26^{64})$.
2. If we think that best Eve can do is $O(n26^n)$ then we take $n = 80$, so Eve needs $O(80 \times 26^{80})$.

The $O(n \times 26^n)$ cracking **does not** show that Matrix Cipher is insecure, but it still is very important: Alice and Bob must increase their parameters. That is already a win since it makes life harder for Alice and Bob.

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Proofs rely on limiting what Eve can do, and hence do not work if Eve does something else.

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3. So this looks like a strong cipher. Is it crackable?

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3. Eve will have old messages and what they decoded to.

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5. Eve knows that $(3, 9) = M(13, 24)$.

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4. We will do this in the next slide packet.