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The Birthday Paradox

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- Number of ways to put balls into boxes so that no box has ≥ 2 balls: $n(n-1)\cdots(n-m+1)$

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Hence we seek

$$\frac{n(n-1)(n-2)\cdots(n-m+1)}{n^m}$$

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$$\sim e^{-m^2/2n}$$

Real Numbers!

If $m < n$ and you put m balls in n boxes at random then prob that ≥ 2 balls in same box is approx:

$$1 - e^{-m^2/2n}$$

To get this $> \frac{1}{2}$ need $1 - e^{-m^2/2n} > \frac{1}{2}$

$$e^{-m^2/2n} < \frac{1}{2}$$

$$-\frac{m^2}{2n} < \ln(0.5) \sim -0.7$$

$$\frac{m^2}{2n} > 0.7$$

$$m > (1.4n)^{1/2}$$

Real Numbers!

If $m > (1.4n)^{1/2}$ and you put m balls in n boxes at random then prob that ≥ 2 balls in same box is over $\frac{1}{2}$.

$n = 365$.
$m = \lceil (1.4n)^{1/2} \rceil = 23$

Birthday Paradox: If there are 23 people in a room then prob two have the same birthday is $> \frac{1}{2}$.

How We Use: If $\sim n^{1/2}$ balls put into n boxes then prob 2 in same box is large.

Alternative Proof

Prob balls i, j in same box is $\frac{n}{n^2} = \frac{1}{n}$.
Prob balls i, j NOT in same box is $1 - \frac{1}{n}$.

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Prob NO pair is in same box
$< (1 - \frac{1}{n})^{\binom{m}{2}} \sim (e^{-1/n})^{m^2/2} \sim e^{-m^2/2n}$.
Prob SOME pair is in same box $> 1 - e^{-m^2/2n}$.
Same as before.

Three Balls in a Box

Prob balls i, j, k in same box is $\frac{n}{n^3} = \frac{1}{n^2}$.
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Prob NO triple is in same box
$\sim (1 - \frac{1}{n^2})^{\binom{m}{3}} \sim (1 - \frac{1}{n^2})^{m^3/6} \sim e^{-m^3/6n^2}$
Prob SOME triple is in same box: APPROX $1 - e^{-m^3/6n^2}$

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If $m < n$ and you put m balls in n boxes at random then prob that ≥ 3 balls in same box is approx:

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$$m^3 > 4.2n^2$$

$$m > (4.2)^{1/3} \times n^{2/3}$$

Birthday Paradox: $n = 365$ then need $m \geq 82$. SO if 82 people in a room prob is $> \frac{1}{2}$ that three have same bday!

How We Use: If $\sim n^{2/3}$ balls put into n boxes then prob 3 in same box is large.

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