#### BILL, RECORD LECTURE!!!!

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# Reminder Types of Attacks

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For all of these attacks. Eve's goal is to find out something about the plaintext she did not already know.

Finding out what was sent is not the only measure of success.

# Learning With Errors: Private Key

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#### Generally:

$$(k_1,\ldots,k_n)\cdot(r_1,\ldots,r_n)=k_1\times r_1+\cdots+k_n\times r_n.$$

We will always be doing this Mod p.

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- Would use a bigger mod and a longer equation in real life.
- ► This cipher only allows transmitting one bit.

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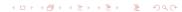
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Number of possibilities for key  $(k_1, k_2, k_3, k_4)$  is now  $191^3$ . If sees more messages can cut down search space to one possibility.



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That is **too** sharp. Instead we will do distinction between:

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We are doing it in a way that is **not used** but **better for education**.

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- $\bullet$   $e \in \{-1,0,1\}$ . In real system  $e \in \{-\gamma,\ldots,\gamma\}$ ,  $\gamma$  a param.
- We picked 50 as our big number. In real system use  $\sim \frac{p}{4}$ .

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We will now use  $\vec{r}$  for a random vector of length n.

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Is this a good cipher? Easy to use? Secure? Discuss.

## Private Key LWE Cipher: Pick $\gamma$ so Works

▶ If b = 0 then Bob compares C to C + e. Diff:  $e \in \{-\gamma, \dots, \gamma\}$ .

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- ▶ If b=1 then Bob compares C to  $C+e+\frac{p}{4}$ . Diff:  $e+\frac{p}{4} \in \{-\gamma+\frac{p}{4},\ldots,\gamma+\frac{p}{4}\}.$

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Need these intervals are disjoint.

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- ▶ If b = 1 then Bob compares C to  $C + e + \frac{p}{4}$ . Diff:  $e + \frac{p}{4} \in \{-\gamma + \frac{p}{4}, \dots, \gamma + \frac{p}{4}\}$ .

Need these intervals are disjoint. Two intervals mod p are disjoint iff when you shift them they are disjoint. We want to shift them to avoid wrap around.

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(We will go into **why** LWE is thought to be hard when we do LWE-public, which won't be for a while.)



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- 1. A problem that plagues complexity theory is that a problem can have a bad worst-case but a reasonable average-case.
- 2. For LWE this is NOT an issue.
- 3. Hence the assumption that LWE is hard for worst case already gives you hard for avg case.

#### **BILL, STOP RECORDING LECTURE!!!!**

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