BILL, RECORD LECTURE!!!!

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Public Key Crypto: Math Needed and Diffie-Hellman

Private-Key Ciphers

What do the following all have in common?

1. Shift Cipher
2. Affine Cipher
3. Vig Cipher
4. General Sub
5. General 2-char sub
6. Matrix Cipher
7. One-time Pad
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Can Alice and Bob establish a key without meeting?
Yes! And that is the **key** to public-**key** cryptography.

General Philosophy

A good crypto system is such that:

1. The computational task to **encrypt** and **decrypt** is **easy.**
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3. Can use hardness assumptions (e.g. factoring is hard).

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Upshot For numeric problems length is $\lg n$. Encryption requires:

- Alice and Bob can Enc and Dec in time $\le (\log n)^{O(1)}$.
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What Counts We count math operations as taking 1 step. This could be an issue with enormous numbers. We will work with mods so not a problem.

Math Needed for Both Diffie-Hellman and RSA

Notation

Let p be a prime.

1. $\mathbb{Z}_p$ is the numbers $\{0, \ldots, p-1\}$ with mod add and mult.
2. $\mathbb{Z}_p^*$ is the numbers $\{1, \ldots, p-1\}$ with mod mult.

Convention By **prime** we will always mean a large prime, so in particular, NOT 2. Hence we can assume $\frac{p-1}{2}$ is in $\mathbb{N}$.

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Problem Given a, n, p find $a^n \pmod{p}$

Even though we use p and p is always prime, our algorithm works for any natural p.

Exponentiation Mod p: First Attempt

Problem Given a, n, p find $a^n \pmod p$

1. $x_0 = a^0 = 1$
2. For $i = 1$ to n, $x_i = ax_{i-1} \pmod p$
3. Let $x = x_n$
4. Output x

Is this a good idea?

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Answer $\sim n$.

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Want $3^{64} \pmod{101}$. All math is mod 101.

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So in 6 steps we got the answer!

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Discuss What if n is **not a power of 2?**

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Where $L = \lfloor \lg(n) \rfloor$ and $n_i \in \{0, 1\}$.

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Where $L = \lfloor \lg(n) \rfloor$ and $n_i \in \{0, 1\}$.
Note that L is one less than the number of bits needed for n.

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Example on next page

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Point: Step 2 took $< \lg(265)$ steps since base-2 rep had few 1's.

Generators and Discrete Logarithms

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3 is a **generator** for $\mathbb{Z}_7^*$.
Definition: If p is a prime and $\{g^1, \ldots, g^{p-1}\} = \{1, \ldots, p-1\}$ then g is a **generator** for $\mathbb{Z}_p^*$.

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Discuss the following with your neighbor:

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$3^x \equiv 92$ easy. $3^x \equiv 93$ Not known how hard.

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2. So finding x such that $g^x \equiv p - g^a \equiv -g^a$ is as easy as g^a.

$$x = \frac{p-1}{2} + a : \qquad g^{\frac{p-1}{2}+a} = g^{\frac{p-1}{2}} g^a \equiv -g^a$$

Discrete Log-Example: $3^x \equiv 93 \pmod{101}$

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Is there a trick for $g^x \equiv 93 \pmod{101}$? Not that I know of.

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- A **bad** alg would be time $p^{O(1)}$.
- If an algorithm is in time (say) $p^{1/10}$ still not efficient but will force Alice and Bob to up their game.

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Good Candidate for a hard problem for Eve.

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5. **Fact:** If computers do DL much better (e.g., $O(p^{1/10})$) then Alice and Bob can increase size of p and be fine. Still, Eve has made them work harder.

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So Kunal and Bill agree.

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Bill Since lots of money is being put into it, if it does not work they won't have the excuse that other technologies have of not having been tried.

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His opinion is on the next slide.

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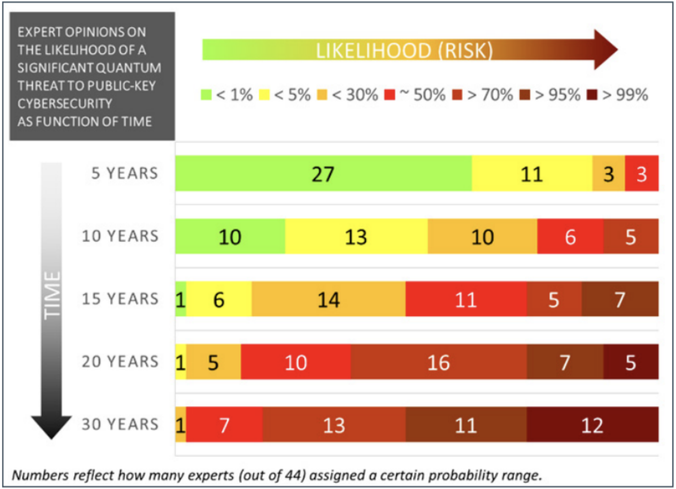
Expert Opinion As a Chart



Expert Opinions on the Technical Realization of Quantum Computers

Expert Opinion As a Paper

See also this paper: `qtime.pdf`

Discrete Log-General

Definition Let p be a prime and g be a generator mod p.
The **Discrete Log Problem:**
Given $a \in \{1, \ldots, p\}$, find x such that $g^x \equiv a \pmod{p}$. We call this $DL_{p,g}(a)$.

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4. **Tradeoff:** By restricting a we are cutting down search space for Eve. Even so, in this case we need to since she REALLY can recognize when DL is easy.

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Do we have this?

No. But we'll come close.

Convention

For the rest of the slides on **Diffie-Hellman Key Exchange** there will always be a prime p that we are considering.

ALL math done from that point on is mod p.

ALL numbers are in $\{1, \ldots, p-1\}$.

Finding Generators

Finding Gens; How Many Gens Are There?

Problem Given p, find g such that

- g generates $\mathbb{Z}_p^*$.
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Hence if you just look for a gen you will find one soon.

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PRO You will find a gen fairly soon.

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Bad! Recall $(\log p)^{O(1)}$ is fast, $O(p)$ is slow.

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Theorem: If g is **not** a generator then there exists x that (1) x divides $p-1$, (2) $x \neq p-1$, and (3) $g^x \equiv 1$.

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Borrow Sajjad's Quantum Computer?

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2. Factor $p-1$. Let F be the set of its factors except $p-1$. That's EASY: $F=\{2,q\}$.
3. For $g=\frac{p}{3}$ to $\frac{2p}{3}$:
   Compute g^x for all $x \in F$. If any $=1$ then g NOT generator. If none are 1 then output g and stop.

Is this a good algorithm?
PRO Every iteration does $O(\log p)$ operations.
CON: Need both p and $\frac{p-1}{2}$ are primes.

Finding Gens: Third Attempt

Idea: Pick p such that $p-1=2q$ where q is prime.
Given prime p, find a gen for $\mathbb{Z}_p^*$

1. Input p a prime such that $p-1=2q$ where q is prime. (We later explore how we can find such a prime.)
2. Factor $p-1$. Let F be the set of its factors except $p-1$. That's EASY: $F=\{2,q\}$.
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Is this a good algorithm?
PRO Every iteration does $O(\log p)$ operations.
CON: Need both p and $\frac{p-1}{2}$ are primes.
CAVEAT We need to pick certain kinds of primes. **Can** do that!

BILL, STOP RECORDING LECTURE!!!!

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