HW 02 Some Solutions

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Hence $\mathrm{COL}(a_{i_1}, a_{i_2}) = \mathrm{COL}(a_{j_1}, a_{j_2})$.

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Could we have left a_1 in?
We have:

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Can check it satisfies condition.
Easily seen to not be homog.

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$x \in X - M$ maps to $(\{p, q\}, r)$ with $\mathrm{COL}(x, r) = \mathrm{COL}(p, q)$.

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Finish on next slide.

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$$|X - M| \le \left| M \times \binom{M}{2} \right| \le \frac{|M|^3}{2} \le \frac{f(n)^3}{2}.$$

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Recall that $|M| \leq f(n)$ so $|X - M| \geq n - f(n)$.

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Recall that $|M| \leq f(n)$ so $|X - M| \geq n - f(n)$.

$$n - f(n) \leq |X - M| \leq \left| M \times \binom{M}{2} \right| \leq \frac{|M|^3}{2} \leq \frac{f(n)^3}{2}.$$

We seek a contradiction. $f(n) = n^{1/3}$ will work.

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Let $\mathrm{COL}\colon \binom{N}{2} \rightarrow$ R be defined by $\mathrm{COL}(i,j) = |p_i - p_j|$.
The number of reals in the image of the colorings is countable so we can apply Can Ramsey. When we apply it we find that there is a set $H \subseteq$ N, $|H| = \infty$ that is either homog, max-homog, min-homog, or rainb. We show H rainb, so all distances different.

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Three cases: homog, min-homog, max-homog.

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Exercise: cannot have 4 points in the plane with all distances the same.

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So $p_2, p_3, \ldots$ are on a circle centered at p_1.

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But two circles with diff centers intersect in at most 2 points.
Contradiction.

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$(\forall i, j \le 4)[|p_4 - p_i| = |p_4 - p_j|]$.
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So p_1, p_2, p_3 are on a circle centered at p_4 and p_5.

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End of Proof of Theorem