Homework 09
Morally Due Tue April 12 at 3:30PM. Dead Cat April 14 at 3:30
WARNING: THE HW IS FOUR PAGES LONG

1. (0 points) What is your name? Write it clearly. When is the take-home final due?

2. (35 points)

   (a) (0 points and it won't help you on the other parts, but do it to give celebrate Morgan's proof) On the slides website is an entry *Morgan's Proof that* $PH(1) \le 7$. Read it. If you find a mistake in it, email Bill. (Note that if you do it will be MY ERROR in transcribing the proof to my format and style, and NOT Morgan's error.)

   (b) (15 points) Present a 2-coloring of $\binom{\{1,\ldots,6\}}{2}$ with no large homog set of size ≥ 3.

   (c) (20 points) Present a 2-coloring of $\binom{\{3,\ldots,9\}}{2}$ with no large homog set (You may or may not need to write a program to find it. That last statement may or may not be a tautology.)

   (d) (Extra Credit) Present a 2-coloring of $\binom{\{3,\ldots,x\}}{2}$ with no large homog set for $x = 10, 11, \ldots$ going as high as you can manage with your computing power. (You might need to write a program to find it).

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3. (35 points)

   (a) (0 points but you need to do this) Read my slides on the prob method being used to get asymptotic lower bounds on $R(k)$. The version you saw in class had some issues, which Liam pointed out to me (yes LIAM, not LIAM's FRIEND nor PERSON WHO SITS x behind and y to the left of Liam) and which, with his help, I fixed.

   (b) (35 points) Fill in the following sentence and prove your result using the Prob Method.

   Fix c, k and think of them as small. There is a graph on n vertices where n is BLANK(c, k) and a c-coloring of $\binom{[n]}{2}$ such that there is NO homog set of size k.

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4. (30 points)

Def Let $G = (V, E)$ be a graph. $D \subseteq V$ is a *dominating set (DS)* if

$$(\forall v \in V)[v \in D \vee (\exists y \in D)[(x, y) \in E].$$

Every graph has a DS of size n: $D = V$. We do better!

You will prove: *There exists a function α such that,*

a) For all $d \in \mathsf{N}$, $0 < \alpha(d) < 1$.

b) The function $\alpha(d)$ is DECREASING.

c) For every graph with min degree $\geq d$ there is a dominating set of size $\leq \alpha(d)n$.

On the next page we will state the theorem with the function α and sketch the proof. YOU will fill in the details and find a function α that works. We guide this with a series of embedded questions.

Thm There exists a function α such that the following hold:

- α maps N to $[0, 1]$.
- α is strictly DECREASING. Hence, for all $d \ge 1$, $\alpha(d) < 1$.
- If $G = (V, E)$ is a graph on n vertices with min degree $\ge d$ then G has a dominating set of size $\le \alpha(d)n$.

Pf Let p be a probability to be determined by YOU later.

Pick $X \subseteq V$ as follows: For every $v \in V$ choose v with probability p.

QUESTION 1 What is $E(|X|)$? It will be a function of n, p.

Let $Y \subseteq V - X$ be the vertices that DO NOT have an edge to an element of X. Formally

$$Y = \{y \in V - X : (\forall x \in X)[(x, y) \notin E].$$

QUESTION 2 Give an upper bound on $E(|Y|)$. It will be a function of n, d, p. Note that $X \cup Y$ is a dominating set. We later pick p so that $|X \cup Y|$ is small.

QUESTION 3 What is $E(|X \cup Y|)$? (Hint: This is very easy by the linearity of expectation.)

QUESTION 4 Pick p to make $E(|X \cup Y|)$ smaller than n (Hint: Find an upper bound on $E(|X \cup Y|)$ and minimize that bound. Use that $(1 - p) \le e^{-p}$.)

QUESTION 5 State and proof a theorem of the form:

Thm If $G = (V, E)$ is a graph on n vertices with min degree $\ge d$ then G has a dominating set of size $\le \alpha(d)n$.

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5. (EXTRA CREDIT) Nash-Williams paper that has a proof of the Kruskal Tree Theorem is on the slides page. The theorem they state is different from the KTT that I stated, though mine can be derived from theirs.

   (a) READ the Nash-Williams Paper.

   (b) DEFINE what a homeomorphism from T to T' is where T and T' are trees. Give examples.

   (c) Prove the Kruskal Tree theorem in your own words. You can assume Lemma 1 and Lemma 2 in the paper and do not need to reprove them. In your proof pay special attention to the part I messed up in class which I recap now:

   Let T be a tree with immediate subtrees $\{T_1, \ldots, T_i\}$.

   Let S be a tree with immediate subtrees $\{S_1, \ldots, S_j\}$.

   Assume that

$$\{T_1, \ldots, T_i\} \preceq \{S_1, \ldots, S_j\}.$$

   Then $T \preceq S$.

   (d) Prove that from the KTT presented in the NW paper, one can derive my KTT.