$LR_2(2) \le \mathbf{13}$

Exposition by William Gasarch

May 20, 2020

Review of $LR_2(k)$

Definition $A \subseteq \mathbb{N}$ is **large** if $|A| > \min(A)$.

Review of $LR_2(k)$

Definition $A \subseteq \mathbb{N}$ is **large** if $|A| > \min(A)$.

Definition $LR_2(k)$ is the least n such that for all 2-colorings of $\binom{\{k,\ldots,n\}}{2}$ there exists a large homog set.

Review of $LR_2(k)$

Definition $A \subseteq \mathbb{N}$ is **large** if $|A| > \min(A)$.

Definition $LR_2(k)$ is the least n such that for all 2-colorings of $\binom{\{k,\ldots,n\}}{2}$ there exists a large homog set.

Definition $LR_2(2)$ is the least n such that for all 2-colorings of $\binom{\{2,\ldots,n\}}{2}$ there exists a large homog set.

$LR_2(2) \le 13$

Let $\mathrm{COL}\colon \binom{\{2,\ldots,13\}}{2} \rightarrow [2]$. We show there is a large homog set.

$LR_2(2) \leq 13$

Let $\mathrm{COL}\colon \binom{\{2,\ldots,13\}}{2} \rightarrow [2]$. We show there is a large homog set.
Note The graph has 12 vertices so every point has degree 11.

$\deg_R(2) \geq 8$

Case 1 $\deg_R(2) \geq 8$. Let the 8 smallest R-neighbors of 2 be $x_1 < \cdots < x_8$.

- There exists $1 \leq i < j \leq 8$ such that $\mathrm{COL}(x_i, x_j) = R$. Large homog set: $\{2, x_i, x_j\}$.
- For all $1 \leq i < j \leq 8$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \leq 7$. Large homog set: $\{x_1, \ldots, x_8\}$.
- For all $1 \leq i < j \leq 8$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \geq 8$. Then $x_8 \geq 15$ which is a contradiction.

$\deg_R(2) = 7$

Case 2 $\deg_R(2) = 7$. Let the 7 smallest R-neighbors of 2 be $x_1 < \cdots < x_7$.

- There exists $1 \le i < j \le 7$ such that $\mathrm{COL}(x_i, x_j) = R$. Large homog set: $\{2, x_i, x_j\}$.
- For all $1 \le i < j \le 7$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \le 6$. Large homog set: $\{x_1, \ldots, x_7\}$.
- For all $1 \le i < j \le 7$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 7$. Note that $\{x_1, \ldots, x_7\} = \{7, 8, 9, 10, 11, 12, 13\}$. Hence the blue neighbors of 2 are $\{3, 4, 5, 6\}$ (1) there exists $3 \le i < j \le 6$ such that (i, j) is B. Large Homog Set: $\{2, i, j\}$. (2) For all $3 \le i < j \le 6$, (i, j) is R. This is a RED K_4 that has 3 as a vertex, so its a large homog set.

$\deg_R(2) = 6$

Case 3 $\deg_R(2) = 6$. Let the 6 smallest R-neighbors of 2 be $x_1 < \cdots < x_6$.

- There exists $1 \le i < j \le 6$ such that $\mathrm{COL}(x_i, x_j) = R$. Large homog set: $\{2, x_i, x_j\}$.
- For all $1 \le i < j \le 6$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \le 5$. Large homog set: $\{x_1, \ldots, x_6\}$.
- For all $1 \le i < j \le 6$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 6$. Note that $\{x_1, \ldots, x_6\} \subseteq \{6, 7, 8, 9, 10, 11, 12, 13\}$. Hence the blue neighbors of 2 contain $\{3, 4, 5\}$. We call the blue neighbors $y_1 = 3 < y_2 = 4 < y_3 = 5 < y_4 < y_5 < y_6$. (1) there exists $1 \le i < j \le 6$ such that (y_i, y_j) is B. Large Homog Set: $\{2, x_i, x_j\}$. (2) For all $1 \le i < j \le 6$, (x_i, x_j) is R. This is a RED K_6 that has 3 as a vertex, so its a large homog set.

$\deg_R(2) \le 5$

Case 4 $\deg_R(2) \le 5$. Then $\deg_B(2) \ge 6$.
If $\deg_B(2) = 6$ use the argument used for $\deg_R(2) = 6$.
If $\deg_B(2) = 7$ use the argument used for $\deg_R(2) = 7$.
If $\deg_B(2) \ge 8$ use the argument used for $\deg_R(2) \ge 8$.