<div align="center">**Computability Theory and Ramsey Theory**</div>

<div align="center">An Exposition by William Gasarch</div>

All of the results in this document are due to Jockusch [2]. For more results in computable combinatorics see the survey by Gasarch [1].

## 1 A Crash Course in Computability Theory

**Notation 1.1**

1. $M_1, M_2, \ldots$ is a standard list of Turing Machines (TMs). You can think of them as all Java programs.

2. We assume that from $e$ we can extract the code for $M_e$.

3. $M_{e,s}(x)$ means that we run $M_e$ for $s$ steps.

4. $M(x) \downarrow = a$ means that $M(x)$ halts and outputs $a$.

5. $M(x) = a$ means that $M(x)$ halts and outputs $a$ (we use the $\downarrow$ when we want to emphasize that $M(x)$ halts).

6. $M(x) \uparrow$ means that $M(x)$ does not halt.

7. A set $A$ is *computable* if there is a TM $M$ such that

$$x \in A \implies M(x) \downarrow = 1$$
$$x \notin A \implies M(x) \downarrow = 0$$

(Older books use the term *recursive* instead of *computable*.)

8. If $M$ is a TM such that on every input $x$, $M(x) \downarrow \in \{0, 1\}$ (so $M$ computes some set) then $L(M) = \{x \colon M(x) = 1\}$ (so $L(M)$ is the set that $M$ computes).

9. A set $A$ is *computably enumerable (c.e.)* if there is a TM $M$ such that

$$x \in A \implies M(x) \downarrow$$
$$x \notin A \implies M(x) \uparrow$$

(Older books use the term *recursively enumerable (r.e.)* instead of *computably enumerable (c.e.).*)

10. $W_e$ is the domain of $M_e$, that is, $W_e = \{x \colon (\exists s)[M_{e,s}(x) \downarrow]$.

11. $W_{e,s} = \{x \colon M_{e,s}(x) \downarrow\}$.

12. A function $f$ is *computable* if there is a TM $M$ such that, for all $x$, $M(x) \downarrow= f(x)$. (Older books use the term *recursive* instead of *computable*.)

Sets are classified in the Arithmetic hierarchy.

**Notation 1.2**

1. $A \in \Sigma_0$ if $A$ is computable.

2. $A \in \Pi_0$ if $A$ is computable.

3. $A \in \Sigma_1$ is there exists $B \in \Pi_0$ such that $A = \{x \colon (\exists y)[(x, y) \in B]\}$.

4. $A \in \Pi_1$ is there exists $B \in \Sigma_0$ such that $A = \{x \colon (\forall y)[(x, y) \in B]\}$.

5. Alternative definition: $A \in \Pi_1$ if $\overline{A} \in \Sigma_1$.

6. For $i \geq 1$ $A \in \Sigma_i$ is there exists $B \in \Pi_{i-1}$ such that $A = \{x \colon (\exists y)[(x, y) \in B]\}$

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**Examples and Facts**

1. $\text{HALT} = \{(e, x) \colon (\exists s)[M_{e,s}(x) \downarrow\} \in \Sigma_1 - \Sigma_0$

2. $W_0, W_1, \ldots$ is a list of all $\Sigma_1$ sets.

3. FIN is the set of all $e$ such that $W_e$ is finite.

$$\text{FIN} = \{e \colon (\exists x)(\forall y, s)[y > x \implies y \notin W_{e,s}\} \in \Sigma_2 - \Pi_2.$$

   (The proof that $\text{FIN} \notin \Pi_2$ is not easy.)

4. INF is the set of all $e$ such that $W_e$ is infinite. $\text{INF} \in \Pi_2 - \Sigma_2$. (The proof that $\text{INF} \notin \Sigma_2$ is not easy.)

5. COF is the set of all $e$ such that $W_e$ is co-finite. We leave it to you to show that $\text{COF} \in \Sigma_3$. (The proof that $\text{COF} \notin \Pi_3$ is not easy.)

6. $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots$.

7. $\Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \cdots$.

8. For all $i \geq 1$, $\Sigma_i$ and $\Pi_i$ are incomparable.

## 2   A Computable Coloring With No Infinite $\Sigma_1$ Homog Set

**Theorem 2.1** *There exists a computable* $\text{COL} \colon \binom{N}{2} \to [2]$ *such that there is no infinite* $\Sigma_1$ *homog set.*

**Proof:**   We use that $W_0, W_1, \ldots$ is a list of all $\Sigma_1$ sets.

We construct computable $\text{COL} \colon \binom{N}{2} \to [2]$ to satisfy the following requirements (NOTE- *requirements* is the most important word in computability theory.)

$$R_e : W_e \text{ infinite} \implies W_e \text{ NOT a homog set .}$$

How can we achieve this? If $x, y, s \in W_e$ and $\mathrm{COL}(x, s) \neq \mathrm{COL}(y, x)$ then $W_e$ is not a homog set. Note that if a set is homog then *every* pair is the same color, so just having *two pairs* be different colors is enough to make the set not homog.

We restate the requirement.

$$R_e : W_e \text{ infinite} \implies (\exists x, y, s \in W_e)[\mathrm{COL}(x, s) \neq \mathrm{COL}(y, s)].$$

Requirement will either be *activated* or *not activated*. Here are some notes about that before doing the formal construction.

1. If $R_e$ is activated then we will associate a set $D_e \subseteq W_e$ with $|D_e| = 2^{e+1}$ to it. When a requirement is in this state we will be working on satisfying it. We won't actually know when it is satisfied; however, we will later prove that it was.

2. If $R_e$ is not activated then we are probably waiting $2^{e+1}$ elements of $W_e$ to appear. If this never happens then $W_e$ is finite so $R_e$ is satisfied. If it does happen then $R_e$ is activated.

3. You might think that we make all of the $D_e$'s disjoint. Alas, this is not possible. But note the following:

   $|D_0| = 2$

   $|D_1| = 4$. Hence $|D_1 - D_0| \geq 2$.

   $|D_2| = 8$. Hence $|D_2 - (D_1 \cup D_0)| \geq 2$.

   and more generally:

   $|D_e| = 2^{e+1}$. Hence $|D_e - \cup_{i=1}^{e-1} D_i| \geq 2^{e+1} - \sum_{i=0}^{e-1} 2^{i+1} = 2$.

## CONSTRUCTION OF COLORING

*Stage 0:* COL is not defined on anything. For all $e$, $R_e$ is not activated.

*Stage s:* We will define $\mathrm{COL}(0,s),\ldots,\mathrm{COL}(s-1,s)$. We will may also activate some requirement and make some progress on requirements that are already activated.

For $e = 0, 1, \ldots, s$:

1. If $R_e$ is not activated then check if there exists $D_e \subseteq W_{e,s} \cap \{0,\ldots,s\}$ such that $|D_e| = 2^{e+1}$. If YES then activated $R_e$ and associate $D_e$ to it.

2. If $R_e$ is activated then let $x, y \in D_e$ be the least numbers that are not in $D_0 \cup \cdots \cup D_{e-1}$. Hence $\mathrm{COL}(x,s)$ and $\mathrm{COL}(y,s)$ have not yet been satisfied. Assume $x < y$. Let:

   - $\mathrm{COL}(x,s) = \mathrm{RED}$

   - $\mathrm{COL}(y,s) = \mathrm{BLUE}$ .

After you to through all all of the $0 \le e \le s$, define all other $\mathrm{COL}(x,s)$ where $0 \le x \le s-1$ that have not been defined by $\mathrm{COL}(x,y) = \mathrm{RED}$ . This is arbitrary. The important things is that ALL $\mathrm{COL}(x,s)$ where $0 \le x \le s-1$ are now defined. This is why COL is computable— at stage $s$ we have defined all $\mathrm{COL}(x,y)$ with $0 \le x < y \le s$.

## END OF CONSTRUCTION

We show that each requirement is eventually satisfied.

For pedagogue we first look at $R_0$.

If $W_0$ is finite then $R_0$ is satisfied.

Assume $W_0$ is infinite. We show that $R_0$ is satisfied. Let $x < y$ be the first two elements that show up in $W_0$. Let $s_0$ be the least number such that $x, y \in W_{0,s_0}$. At state $s_0$, $R_0$ will be activated with $D_0 = \{x,y\}$. Note that, for ALL $s \ge s_0 - 1$:

$\mathrm{COL}(x,s) = \mathrm{RED}$

$\mathrm{COL}(y,s) = \mathrm{BLUE}$

Since $W_0$ is infinite there is SOME $s \geq s_0 + 1$ with $s \in W_e$. Hence $x, y, s \in W_0$ and show that $W_0$ is NOT homogenous.

Can we show $R_1$ is satisfied the same way? Yes but with a caveat- we won't use the first two elements that show up on $W_1$. We'll use the first two elements that show up on $W_1$ that are not in $D_0$. But there is a further caveat which we illustrate with an example.

1. At Stage 100 $R_1$ is activated with $D_1 = \{10, 11, 19, 22\}$. $R_0$ has still not been activated.

2. For $101 \leq s \leq 999$ $R_0$ has still not been activated. Hence when $R_1$ is processed we get:

   (a) $COL(10, s) = RED$

   (b) $COL(11, s) = BLUE$

3. Stage 1000: $R_0$ gets activated with $D_0 = \{11, 111, 299, 788\}$?

4. Let $s \geq 1000$.

   When $R_0$ is processed we get:

   (a) $COL(11, s) = RED$

   (b) $COL(111, s) = BLUE$.

   When $R_1$ is processed we get:

   (a) $COL(10, s) = RED$

   (b) $COL(19, s) = BLUE$.

Lets just look at $R_1$. If $W_1$ is infinite then there exists an $s \geq 100$ such that $s \in W_1$. If $100 \leq s \leq 999$ then we $R_1$ is satisfied by using $\{10, 11, s\}$. If $s \geq 1000$ then we $R_1$ is satisfied by using $\{10, 19, s\}$. Note that all that matters is that once $R_1$ is activated, $R_1$ will be satisfied and it does not matter what $R_0$ is doing.

We now prove that the all requirements are satisfied.

**Claim:** Let $e \in \mathsf{N}$. Then $W_e$ is satisfied.

**Proof of Claim:**

If $W_e$ is finite then $R_e$ is satisfied. So we assume $W_e$ is infinite. Let $s_e$ be the least number such that

- $e \leq s_e$.

- $|W_{e,s_e} \cap \{0, \ldots, s_e\}| \geq 2^{e+1}$.

Then $R_e$ will be activated at stage $s_e$ and $D_e$ will be created.

For every stage $s \geq s_e$, when $R_e$ is processed there will be $x < y \in D_e$ such that

- $\mathrm{COL}(x, s) = \mathrm{RED}$

- $\mathrm{COL}(y, s) = \mathrm{BLUE}$ .

We know that $x, y \in W_e$ but we know nothing about $s$. However, $W_e$ is infinite. Let $s$ be the least element of

$$\{s_e, s_e + 1, s_e + 2, \ldots\}$$

that is in $W_e$. At stage $s$ we will set $\mathrm{COL}(x, s) = \mathrm{RED}$ and $\mathrm{COL}(y, s) = \mathrm{BLUE}$ . Since $x, y, s \in W_e$, requirement $R_e$ is satisfied.

**End of Proof of Claim**

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## 3 Every Computable Coloring has an Infinite $\Sigma_3$ Homog set

Take the standard proof of the infinite 2-ary Ramsey Theorem. Let COL be the given coloring of $\binom{\mathsf{N}}{2}$. Assume COL is computable.

The function $\mathrm{COL}'$ from $\mathsf{N}$ to $\{R, B\}$ can be computed by asking $\Pi_2$ questions. Hence we say informally $\mathrm{COL}' \leq_{\mathrm{T}} \Pi_2$. One can show that using this all three sets: $R$, $B$, and $DEAD$ are $\Sigma_3$.

We now have a subtle point. If all we want to know is the complexity of a homog set we can say that ONE OF $R$ or $B$ is infinite, hence there IS a $\Sigma_3$-homog set. And this is the answer we will give. But notice that we do not know which of $R$ or $B$ is the homog set. That would require a $\Sigma_4$-question.

Can we do better? YES! See the next section.

## 4 Every Computable Coloring has an Infinite $\Pi_2$ Homog set

We obtain this with a modification of the usual proof of Ramsey's theorem. the key is that we don't really toss things out- we guess on what the colors are and change our mind.

**Theorem 4.1** *For every computable coloring* $\mathrm{COL} \colon \binom{\mathsf{N}}{2} \to [2]$ *there is an infinite $\Pi_2$ homog set.*

**Proof:**

We are given computable $\mathrm{COL} \colon \binom{\mathsf{N}}{2} \to [2]$.

CONSTRUCTION of $x_1, x_2, \ldots$ and $c_1, c_2, \ldots$.

NOTE: at the end of stage $s$ we might have $x_1, \ldots, x_i$ defined where $i < s$. We will not try to keep track of how big $i$ is. Also, we may have at stage (say) 1000 a sequence of length 50, and then at stage 1001 have a sequence of length only 25. The sequence will grow eventually but do so in fits and starts.

$$x_1 = \quad 1$$

$$c_1 = \quad \text{RED}\ \text{We are guessing. We might change our mind later}$$

Let $s \geq 2$, and assume that $x_1, \ldots, x_{s-1}$ and $c_1, \ldots, c_{s-1}$ are defined.

1. Ask HALT

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*Does there exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq s-1$, $\mathrm{COL}(x_i, x) = c_i$?*

2. If YES then (using that $\mathrm{COL}$ is computable) find the least such $x$.

$$x_i = \quad x$$

$$c_i = \quad \text{RED We are guessing. We might change our mind later}$$

We have implicitly tossed out all of the numbers between $x_{i-1}$ and $x_i$.

3. If NO then we ask $\mathrm{HALT}$ how far back we can go. More rigorously we ask the following sequence of questions until we get a YES.

   - *Does there exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq s-2$, $\mathrm{COL}(x_i, x) = c_i$)?*

   - *Does there exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq s-3$, $\mathrm{COL}(x_i, x) = c_i$)?*

   - $\vdots$

   - *Does there exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq 2$, $\mathrm{COL}(x_i, x) = c_i$)?*

   - *Does there exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq 1$, $\mathrm{COL}(x_i, x) = c_i$)?*

(One of these must be a YES since (1) if $c_1 = RED$ and there are NO red edges coming out of $x_1$ then there must be an infinite number of BLUE edges, and (2) if $c_1$=BLUE its because there are only a finite number of RED edges coming out of $x_1$ so there are an infinite number of BLUE edges. Let $i_0$ be such that *There exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq i_0$, $\mathrm{COL}(x_i, x) = c_i$*) Do the following:

   (a) Change the color of $c_{i+1}$. (We will later see that this change must have been from RED to BLUE .

   (b) Wipe out $x_{i+2}, \ldots, x_{s-1}$.

   (c) Search for the $x \geq x_{s-1}$ that the question asked says exist.

(d) $x_{i+2}$ is now $x$.

(e) $c_{i+2}$ is now RED .

END OF CONSTRUCTION of $x_1, x_2 \ldots$ and $c_1, c_2, \ldots$.

We need to show that there is a $\Pi_2$ homog set.

Let $X$ be the set of $x_i$ that are put on the board and stay on the board.

Let $R$ be the set of $x_i \in X$ whose final color is RED .

**Claim 1:** Once a number turns from RED to BLUE it can't go back to RED again.

**Proof:**

If a number is turned BLUE its because there are only a finite number of RED edges coming out of it. Hence there must be an infinite number of BLUE edges coming out of it. Hence it will never change color (though it may be tossed out).

**End of Proof**

**Claim 1:** $X, R \in \Pi_2$.

**Proof:**

We show that $\overline{X} \in \Sigma_2$. In order to NOT be in $X$ you must have, at some point in the construction, been tossed out.

$$\overline{X} = \{x \colon (\exists x)[\text{ at stage } s \text{ of the construction } x \text{ was tossed out }]\}.$$

Note that the condition is computable-in-$\mathrm{HALT}$. Hence $\overline{X}$ is c.e.-in-$\mathrm{HALT}$. It is known that if a set is c.e.-in-$\mathrm{HALT}$ then it is in $\Sigma_2$. Hence $\overline{X} \in \Sigma_2$.

We show that $\overline{R} \in \Sigma_2$. In order to NOT be in $R$ you must have to either NOT be in $X$ or have been turned blue. Note that once you turn at some point in the construction, been tossed out.

$$\overline{R} = \overline{X} \cup \{x \colon (\exists x)[\text{ at stage } s \text{ of the construction } x \text{ was turned BLUE}]\}.$$

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Note that the condition is computable-in-HALT. Hence $\overline{R}$ is c.e.-in-HALT. so $\overline{R} \in \Sigma_2$.

**End of Proof**

We have shown $X, R$ are $\Pi_2$ but have not shown that $B$ is- and in fact $B$ might not be. But we show that $B$ is $\Pi_2$ when we need it to be.

There are two cases:

1. If $R$ is infinite then $R$ is an infinite homog set that is $\Pi_2$.

2. If $R$ is finite then $B$ is $X$ minus a finite number of elements. Since $X$ is $\Pi_2$, $B$ is $\Pi_2$.

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**References**

[1] W. Gasarch. A survey of recursive combinatorics. In Ershov, Goncharov, Nerode, and Remmel, editors, *Handbook of Recursive Algebra*, pages 1041–1171. North Holland, 1997. `http://www.cs.umd.edu/~gasarch/papers/papers.html`.

[2] C. Jockusch. Ramsey's theorem and recursion theory. *Journal of Symbolic Logic*, 37(2):268–280, 1972.
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