Asy Lower Bounds on Ramsey Numbers

Exposition by William Gasarch

Summary Of Talk

- We obtain asy lower bounds on $R(k)$.

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- We obtain asy lower bounds on $R(k)$.
- We then use the method to do other things, outside of Ramsey Theory.

Recall Upper Bound on Ramsey Numbers

We know that

$$R(k) \le 2^{2k-1}$$

One can also get

$$R(k) \le \binom{2k-2}{k-1} \sim \frac{2^{2k}}{\sqrt{k}}$$

We want to find lower bounds

PROBLEM We want to find a coloring of the edges of K_n w/o a mono K_k. for some $n = f(k)$.

A Lower Bound

Theorem $R(k) \geq (k-1)^2$.
Proof
Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:
First partition $[(k-1)^2]$ into $k-1$ groups of $k-1$ each.

$$COL(x,y) = \begin{cases} \text{RED} & \text{if } x, y \text{ are in same } V_i \\ \text{BLUE} & \text{if } x, y \text{ are in different } V_i \end{cases} \quad (1)$$

Look at any k vertices.

- They can't all be in one V_i, so it can't have RED K_k.
- They can't all be in different V_i, so it can't have BLUE K_k.

Recap

$$(k-1)^2 \leq R(k) \leq 2^{2k-1}$$

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WRONG QUESTION I only need show that such a coloring exists.

Pick a coloring at Random!

Numb of colorings: $2^{\binom{n}{2}}$.
Numb of colorings: that have mono K_k is bounded by

$$\binom{n}{k} \times 2 \times 2^{\binom{n}{2}-\binom{k}{2}}$$

Prob that a random 2-coloring HAS a homog set is bounded by

$$\frac{\binom{n}{k} \times 2 \times 2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}}} \le \frac{\binom{n}{k} \times 2}{2^{\binom{k}{2}}} \le \frac{n^k}{k!2^{k^2/2}}$$

Want n large and $\frac{n^k}{k!2^{k^2/2}} < 1$.

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We take $n = k2^{k/2}$ to get:

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Slightly better lower bounds are known, but still roughly $k2^{k/2}$.

DISTINCT DIFF SETS

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Given n try to find a set $A \subseteq \{1, \ldots, n\}$ such that ALL of the differences of elements of A are DISTINCT.

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Can we do better?
STUDENTS break into small groups and try to either do better OR show that you best you can do is $O(\log n)$.

An Approach

Let a be a number to be determined.

Pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a.

What is the probability that all of the diffs in A are distinct?

We hope the prob is strictly GREATER THAN 0.

KEY: If the prob is strictly greater than 0 then there must be SOME set of a elements where all of the diffs are distinct.

Determining the Prob

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are distinct?

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We hope the Prob is strictly LESS THAN 1.

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We only need to show that the prob is LESS THAN 1.

Review a Little Bit of Combinatorics

The number of ways to CHOOSE y elements out of x elements is

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}.$$

Determining the Prob I

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Two ways to create a set with a diff repeated:

Way One:

- Pick $x < y$. There are $\binom{n}{2} \le n^2$ ways to do that.
- Pick diff d such that $x + d \ne y$, $x + d \le n$, $y + d \le n$. Can do $\le n$ ways. Put $x, y, x + d, y + d$ into A.
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Way Two: Pick $x < y$. Let $d = y - x$ (so we do NOT pick d). Put $x, y = x + d, y + d$ into A. Pick $a - 3$ more elements out of the $n - 3$ left.

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Number of ways to do this is $\le n^2 \times \binom{n-3}{a-3}$.

Determining the Prob II

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a then a bound on the probability that all of the diffs in A are NOT distinct is

$$\frac{n^3 \times \binom{n-4}{a-4} + n^2 \times \binom{n-3}{a-3}}{\binom{n}{a}} = \frac{n^3 \times \binom{n-4}{a-4}}{\binom{n}{a}} + \frac{n^2 \times \binom{n-3}{a-3}}{\binom{n}{a}}$$

$$= \frac{n^3 a(a-1)(a-2)(a-3)}{n(n-1)(n-2)(n-3)} + \frac{n^2 a(a-1)(a-2)}{n(n-1)(n-2)}$$

$$\leq \frac{32a^4}{n} \text{ Need some Elem Algebra and uses } n \geq 5.$$

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UPSHOT: For all $n \geq 5$ there exists a all-diff-distinct subset of $\{1, \ldots, n\}$ of size roughly $n^{1/4}$.

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- Caveat: If the Prob Proof has high prob of getting the object, then seems constructive. If all you prove is nonzero, than maybe not.

Actually Can Do Better

- With a maximal set argument can do $\Omega(n^{1/3})$.
- Better is known: $\Omega(n^{1/2})$ which is optimal

SUM FREE SET PROBLEM

Exposition by William Gasarch

Sum Free Set Problem

A More Sophisticated Use of Prob Method.
Definition: A set of numbers A is *sum free* if there is NO $x, y, z \in A$ such that $x + y = z$.

Example: Let $y_1, \ldots, y_m \in (1/3, 2/3)$ (so they are all between $1/3$ and $2/3$). Note that $y_i + y_j > 2/3$, hence $y_i + y_j \notin \{y_1, \ldots, y_m\}$.

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Proof: STUDENTS DO THIS. ITS EASY.
Example: Let $A = \{y_1, \ldots, y_m\}$ all have fractional part in $(1/3, 2/3)$. A is sum free by above Lemma.

QUESTION

Given $x_1, \ldots, x_n \in R$ does there exist a LARGE sum-free subset? How Large?

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VOTE:

1. There is a sumfree set of size roughly $n/3$.
2. There is a sumfree set of size roughly $\sqrt{n}$.
3. There is a sumfree set of size roughly $\log n$.

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STUDENTS - WORK ON THIS IN GROUPS.

SUM SET PROBLEM

Theorem For all $\epsilon > 0$, for all A that are a set of n real numbers, there is a sum-free subset of A of size $(1/3 - \epsilon)n$.
Proof: Let L be LESS than everything in A and U be BIGGER than everything in A. We will make $U - L$ LARGE later.
For $a \in [L, U]$ let

$$B_a = \{x \in A : \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

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$$B_a = \{x \in A : \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

For all a, B_a is sum-free by Lemma above.
SO we need an a such that B_a is LARGE.

How Big IS B_a?

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Let $x \in A$.

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$$\Pr_{a \in [L,U]}(\mathrm{frac}(ax) \in (1/3, 2/3))$$

We take $U - L$ large enough so that this prob is $\ge (1/3 - \epsilon)$.

$$E(|B_a|) = \sum_{x \in A} \Pr_{a \in [L,U]}(\mathrm{frac}(ax) \in (1/3, 2/3))$$
$$= \sum_{x \in A} (1/3 - \epsilon)$$
$$= (1/3 - \epsilon)n.$$

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So THERE EXISTS an a such that $|B_a| \ge (1/3 - \epsilon)n$.
What is a? I DON"T KNOW AND I DON"T CARE!
End of Proof

Turan's Theorem

Exposition by William Gasarch

Turan's Theorem

Theorem If $G = (V, E)$ is a graph, $|V| = n$, and $|E| = e$, then G has an ind set of size at least

$$\frac{n}{\frac{2e}{n} + 1}.$$

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We proof this using Probability, but first need a lemma.

Lemma

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Proof: Try to count the edges by summing the degrees at each vertex. This counts every edge TWICE.

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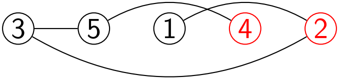
Example:



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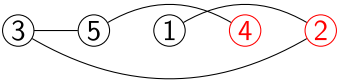

The set of vertices that have NO edges coming out on the right form an Ind Set. Call this set I.

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WRONG QUESTION!

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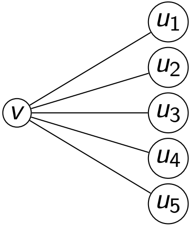
What is the EXPECTED VALUE of the size of I.

(NOTE- we permuted the vertices RANDOMLY)

What is Prob $v \in I$

Let $v \in V$. What is prob that $v \in I$



v has degree d_v. How many ways can v and its vertices be laid out: $(d_v + 1)!$. In how many of them is v on the right? $d_v!$.

$$\Pr(v \in I) = \frac{d_v!}{(d_v+1)!} = \frac{1}{d_v+1}.$$

Hence

$$E(|I|) = \sum_{v \in V} \frac{1}{d_v+1}.$$

How Big is this Sum?

Need to find lower bound on

$$\sum_{v \in V} \frac{1}{d_v + 1}.$$

Rephrase

NEW PROBLEM:
Minimize

$$\sum_{v \in V} \frac{1}{x_v + 1}$$

relative to the constraint:

$$\sum_{v \in V} x_v = 2e.$$

KNOWN: This sum is minimized when all of the x_v are $\frac{2e}{|V|} = \frac{2e}{n}$.
So the min the sum can be is

$$\sum_{v \in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$

Recap and Done

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To lower bound $E(|I|)$ we solve a continous problem: minimize $\sum_{v \in V} \frac{1}{x_v+1}$ with constraint $\sum_{v \in V} \frac{1}{x_v+1} = 2e$.

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The min occurs when $(\forall v)[x_v = \frac{2e}{n}]$. Hence

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The min occurs when $(\forall v)[x_v = \frac{2e}{n}]$. Hence

$$E(I) \ge \sum_{v \in V} \frac{1}{x_v + 1} \ge \sum_{v \in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$

END OF THIS TALK/TAKEAWAY

END OF THIS TALK

TAKEAWAY: There are TWO ways (probably more) to show that an object exists using probability.

1. Show that the probability that it exists is NONZERO. Hence there must be some set of random choices that makes it exist. We did this for the distinct-sums problem.
2. You want to show that an object of a size $\geq s$ exists. Show that if you do a probabilistic experiment then you (a) always get the object of the type you want, and (b) the expected size is $\geq s$. Hence again SOME set of random choices produces an object of size $\geq s$.