

# The Square Theorem

**Exposition by William Gasarch**

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# The Square Theorem

**Definition** Let  $G \in \mathbb{N}$  and  $c \in N$ . Let  $\text{COL}: [G] \times [G] \rightarrow [c]$ .

1. A **mono  $L$**  is 3 points

$$(x, y), (x + d, y), (x, y + d)$$

that are all the same color ( $d \geq 1$ ). This is an isosceles  $L$ .

2. A **mono Square** is 4 points

$$(x, y), (x + d, y), (x, y + d), (x + d, y + d)$$

that are all the same color ( $d \geq 1$ ). This is a square.

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3. To prove **The Square Theorem** (about 2-coloring) we need to know that  $L(c)$  exists for a very large  $c$ .
4. More Colors: *For all  $c$  there exists  $G = G(c)$  such that for all  $\text{COL}: [G] \times [G] \rightarrow [c]$  there exists a mono square.* Proof needs a larger  $c'$  for  $L(c')$ .

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Take the  $[H] \times [H]$  grid and tile it with  $3 \times 3$  tiles.

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**This is very typical of VDW-Ramsey Theory: a 2-coloring of BLAH is viewed as a  $X$ -coloring of a different object where  $X$  is quite large.**

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- ▶ Easier to prove it from the Hales-Jewitt Theorem, which we won't be doing.

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Go to Whiteboard for rest of proof.