$\mathrm{PH}(1) \le \mathbf{8}$

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Review of $\mathrm{PH}(k)$

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Def $\mathrm{PH}(k)$ is the least $n \geq$ such that for all 2-colorings of $\binom{\{k,\ldots,n\}}{2}$ there exists a homog set H such that (a) $|H| > \min(H)$ and (b) $|H| \geq 3$.
(PH stands for Paris-Harrington.)

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Note The graph has 8 vertices so every point has degree 7.

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- For all $1 \leq i < j \leq 5$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \leq 4$. Large homog set: $\{x_1, x_2, x_3, x_4, x_5\}$.

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- For all $1 \le i < j \le 5$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 5$. $x_2 \ge 6$, $x_3 \ge 7$, $x_4 \ge 8$, $x_5 \ge 9$.

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- For all $1 \le i < j \le 5$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 5$. $x_2 \ge 6$, $x_3 \ge 7$, $x_4 \ge 8$, $x_5 \ge 9$. Contradiction since we are coloring $\binom{\{1,\ldots,8\}}{2}$.

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Last Case on Next Slide.

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For all $1 \le i < j \le 4$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 4$. Hence $\{x_1, x_2, x_3, x_4\} \subseteq \{4, 5, 6, 7, 8\}$.
Hence B neighbors of 1 are $\{2, 3\}$ and $x \in \{4, 5, 6, 7, 8\}$.

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For all $1 \le i < j \le 4$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 4$. Hence $\{x_1, x_2, x_3, x_4\} \subseteq \{4, 5, 6, 7, 8\}$.
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- If $\mathrm{COL}(2, 3) = B$ then Large homog set is $\{1, 2, 3\}$.
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- If $\mathrm{COL}(3, x) = B$ then Large homog set is $\{1, 3, x\}$.
- If none of he above hold then Large homog set is $\{2, 3, x\}$.

$\deg_{\mathbf{R}}(1) \leq 3$

Case 3 $\deg_{\mathbf{R}}(1) \leq 3$. Then $\deg_{\mathbf{B}}(1) \geq 4$.
If $\deg_{\mathbf{B}}(1) = 4$ use the argument used for $\deg_{\mathbf{R}}(1) = 4$.
If $\deg_{\mathbf{B}}(1) \geq 5$ use the argument used for $\deg_{\mathbf{R}}(1) \geq 5$.

Exact Bound on $\mathrm{PH}(1)$

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