The Infinite a-ary Can Ramsey Thm

William Gasarch-U of MD

Min-Homog, Max-Homog, Rainbow

Def: Let $\mathrm{COL} : \binom{\mathbb{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- V is *homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff *TRUE*.
- V is *min-homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff $a = c$.
- V is *max-homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff $b = d$.
- V is *rainb* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff $a = c$ and $b = d$.

Can Ramsey Thm for $\binom{\mathbb{N}}{2}$: For all $\mathrm{COL} : \binom{\mathbb{N}}{2} \rightarrow \omega$, there exists an infinite set V such that V is homog OR min-homog OR max-homog OR rainb.

Restate So We Can Generalize

Def: Let $\mathrm{COL} : \binom{\mathbb{N}}{2} \to \omega$. Let $V \subseteq \mathbb{N}$. Assume $a_1 < a_2$ and $b_1 < b_2$.

- V is *homog* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff *TRUE*. So $\mathrm{COL}(x, y)$ does not depend on the first or second coordinate. We call this $\emptyset$-homog.
- V is *min-homog* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff $a_1 = b_1$. So $\mathrm{COL}(x, y)$ depend on the first coordinate only. We call this $\{1\}$-homog.
- V is *max-homog* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff $a_2 = b_2$. So $\mathrm{COL}(x, y)$ depend on the second coordinate only. Can call this $\{2\}$-homog.
- V is *rainb* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff $a_1 = b_1$ and $a_2 = b_2$. So $\mathrm{COL}(x, y)$ depend on the first and second coordinate only. Can call this $\{1, 2\}$-homog.

Can Ramsey Thm for $\binom{\mathbb{N}}{2}$: For all $\mathrm{COL} : \binom{\mathbb{N}}{2} \to \omega$, there exists $A \subseteq \{1, 2\}$ and an infinite set V such that V is A-homog.

All 8 Cases For 3-Ary Can Ramsey

$\mathrm{COL} : \binom{\mathbb{N}}{3} \to \omega$. $V \subseteq \mathbb{N}$. $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$.

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Define $\mathrm{COL} : \binom{\mathbb{N}}{3} \rightarrow \omega$ by

$$\mathrm{COL}(x < y < z) = (x, z)$$

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The rest of the cases are similar.

Proofs of 3-ary Can Ramsey

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3. One is Mileti-Style.

Doing these is extra credit on hw02.

a-ary Can Ramsey

I leave it to you to state and prove a-ary Can Ramsey.