Duplicator Spoiler Games
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1 Introduction

Roland Fraïssé [2, 3, 4] proved theorems about logical expressibility using a back-and-forth method. Andrzej Ehrenfeucht [1] formalized Fraïssé's method by invented Duplicator-Spoiler Games[1].

In this exposition we will define Duplicator-Spoiler games for linear orderings and prove the connection to logical expressible for these games. We follow the treatment of [5] which is out of print.

2 Duplicator-Spoiler Games For Linear Orderings

Definition 2.1 A linear ordering $\mathcal{L}$ is defined as a set L paired with an ordering $<$, denoted $\mathcal{L} = (L, <)$, such that

1. $(\forall\ x, y \in L)$ $[x < y$ or $x \geq y$, but not both$]$.
2. $(\forall\ x, y, z \in L)[x < y \wedge y < z \implies x < z]$.

We now define Duplicator-Spoiler Games played with linear orderings.

Definition 2.2 Let $\mathcal{L}_1 = (L_1, <)$, $\mathcal{L}_2 = (L_2, <)$. The *m-round Duplicator-Spoiler Game on* $(\mathcal{L}_1, \mathcal{L}_2)$ is defined as follows.

1. There are two players: the Spoiler and the Duplicator.
2. There are m rounds. During round i $(1 \leq i \leq m)$ the Spoiler selects an element from either set and the Duplicator selects an element from the other set. The element selected from L_1 is called a_i and from L_2, b_i.
3. If $(\forall\ i, j,\ 1 \leq i, j \leq m)$ $[a_i < a_j \iff b_i < b_j]$, then the Duplicator wins. Otherwise, the Spoiler wins.

Definition 2.3 Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear orderings. Let $m \in \mathsf{N}$. If Duplicator wins the m-round Duplicator-Spoiler game on $(\mathcal{L}_1, \mathcal{L}_2)$ then $\mathcal{L}_1$ and $\mathcal{L}_2$ are *m-game equivalent* which we denote $\mathcal{L}_1 \equiv_m^G \mathcal{L}_2$.

Geometrically, we imagine lines being drawn from elements of one set to another, each line representing a pair of inequalities. If two lines cross, the inequalities are not consistent from set to set and the Spoiler wins. Because of this, a line, or round, reduces any game into two new ones.

Convention 2.4 Unless otherwise specified, we will assume the Spoiler and the Duplicator play optimally. In other words, if one player can implement a strategy that wins every time then the player is assumed to implement said strategy.

[1] Duplicator-Spoiler games are also referred to as Ehrenfeucht-Fraïssé games.

Notation 2.5 The following notations will be used throughout.

- Let $\mathcal{F}_m$ be the finite linear ordering of m elements.
- Let ω be the linear ordering $\{1 < 2 < 3 < \cdots\}$. Note that this is equivalent to $\{2 < 4 < \cdots\}$. The base set is unimportant.
- Let ω^* be the linear ordering $\{\cdots < -3 < -2 < -1\}$.
- Let Z be the linear ordering $\{\cdots < -2 < -1 < 0 < 1 < 2 \cdots\}$.
- Let Q be the natural ordering of the rational numbers.
- Let R be the natural ordering of the real numbers.

Definition 2.6 Let $\mathcal{L}_1 = (L_1, <_1), \mathcal{L}_2 = (L_2, <_2)$. The linear ordering $\mathcal{L}_1 + \mathcal{L}_2$ is formed as follows.

We can assume that $L_1 \cap L_2 = \emptyset$ by changing the elements' labels. Let the base set for $\mathcal{L}_1 + \mathcal{L}_2$ be $L_1 \cup L_2$. Let the total order be as follows.

1. $(\forall\ x, y \in L_1)\ [x < y \iff x <_1 y]$.
2. $(\forall\ x, y \in L_2)\ [x < y \iff x <_2 y]$.
3. $(\forall\ x \in L_1)(\forall\ y \in L_2)\ [x < y]$.

BILL-DO EXAMPLES LEADING UP TO N and N+Z.

3 The Connection to Logic

We first define formulas and two notions of the complexity of formulas: *quantifier depth* and *number of free variables.*

We will now define formulas and quantifier depth (qd) rigorously.

Definition 3.1 Our language contains the symbols $\wedge, \vee, \neg, \exists, \forall, =, <$ and variables $x_1, x_2, \ldots$. We may use x, y, z for notational convenience.

1. A variable is *free* if it is not quantified over. When we write (say) $\phi(x)$ the x is a free variable. There may be other variables; however, they are quantified over.
2. An *atomic formula* is any formula of the form $x_i < x_j$ or $x_i = x_j$. If $\phi(x_i, x_j)$ is an atomic formula then $\mathrm{qd}(\phi(x_i, x_j)) = 0$.
3. If $\phi(\vec{x})$ is a formula then $\neg\phi(\vec{x})$ is a formula and $\mathrm{qd}(\neg(\phi(\vec{x}))) = \mathrm{qd}(\phi(\vec{x}))$.
4. If $\gamma(\vec{x})$ and $\theta(\vec{y})$ are formulas then
   - $\gamma(\vec{x}) \wedge \theta(\vec{y})$ is a formula and $\mathrm{qd}(\gamma(\vec{x}) \wedge \theta(\vec{y})) = \max\{\mathrm{qd}(\gamma(\vec{x})), \mathrm{qd}(\theta(\vec{y}))\}$.
   - $\gamma(\vec{x}) \vee \theta(\vec{y})$ is a formula and $\mathrm{qd}(\gamma(\vec{x}) \vee \theta(\vec{y})) = \max\{\mathrm{qd}(\gamma(\vec{x})), \mathrm{qd}(\theta(\vec{y}))\}$,

5. If $\phi(\vec{x}, x)$ is a formula then $(\exists x)[\phi(\vec{x}, x)]$ is a formula and $\mathrm{qd}((\exists x)[\phi(\vec{x}, x)]) = \mathrm{qd}(\phi(\vec{x}, x)) + 1$. Note that $(\exists x)[\phi(\vec{x}, x)]$ has one less free variable then $\phi(\vec{x}, x)$.

Definition 3.2 A *sentence* is a formula with no free variables.

Let ϕ be a sentence like $(\exists x)(\forall y)[x \le y]$. Is this sentence true or false? This is a stupid question: you need to know which linear order is being talked about. The next definition gives a succinct way of saying this.

Definition 3.3 Let $\mathcal{L} = (L, <)$ be a linear ordering.

1. Let ϕ be a sentence. $\mathcal{L} \models \phi$ means that ϕ is true when interpreted in $\mathcal{L}$.
2. $\mathcal{L}_1$ and $\mathcal{L}_2$ are *m-truth-equivalent*, denoted $\mathcal{L}_1 \equiv^T_m L_2$, if, for all ϕ with $\mathrm{qd}(\phi) \le m$

$$\mathcal{L}_1 \models \phi \text{ iff } \mathcal{L}_2 \models \phi.$$

We want to prove the following theorem:

Theorem 3.4 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear orderings. Let $m \in \mathsf{N}$. The following are equivalent:*

1. $\mathcal{L}_1 \equiv^G_m \mathcal{L}_2$.
2. $\mathcal{L}_1 \equiv^T_m \mathcal{L}_2$.

However, this is one of those cases where it is easier to prove a harder theorem. We will (1) extend Duplicator-Spoiler games to the case where some of the moves are already specified, and (2) extend the definition of $\models$ to formulas with parameters. We will then show that those notions are equivalent.

Definition 3.5 Let $\mathcal{L}_1 = (L_1, <)$, $\mathcal{L}_2 = (L_2, <)$, $\vec{a} \in L_1^k$, and $\vec{b} \in L_2^k$. The *m-round Duplicator-Spoiler Game on* $((\mathcal{L}_1, \vec{a}), (\mathcal{L}_2, \vec{b}))$, is defined as follows.

1. There are two players: The Spoiler and The Duplicator.
2. There are m rounds. During round i $(1 \le i \le m)$ Spoiler selects an element from either set and Duplicator selects an element from the other set. The element selected from L_1 is called a_{k+i} and from L_2, b_{k+i}.
3. If $(\forall\ i, j,\ 1 \le i, j \le m + k)$ $[a_i < a_j \iff b_i < b_j]$, then Duplicator wins. Otherwise, Spoiler wins.

Definition 3.6 If Duplicator wins the m-round game then $(\mathcal{L}_1; \vec{a}) \equiv^G_m (\mathcal{L}_2; \vec{b})$.

The game is essentially the same as the original Duplicator-Spoiler game; however, the first k rounds have already been played.

Note the following

Fact 3.7

1. *$(\mathcal{L}_1;\vec{a}) \equiv_0^G (\mathcal{L}_2;\vec{b})$ iff $\vec{a}$ and $\vec{b}$ are of the same order type.*

2. *Assume $(\mathcal{L}_1;\vec{a}) \equiv_{m+1}^G (\mathcal{L}_2;\vec{b})$. If Spoiler plays a and Duplicator's winning response is b then $(\mathcal{L}_1;\vec{a},a) \equiv_m^G (\mathcal{L}_2;\vec{b},b)$.*

Let $\phi(\vec{x})$ be a formula with k free variables. For example, if $k=1$ then $(\exists x)[x \le y]$ would be such a formula. Is this sentence true or false? This is an even stupider question than the one about sentences: you need to know which linear order $\mathcal{L} = (L,<)$ is being talked about AND you need to know the k elements of L that you intend to plug into ϕ. The next definition gives a succinct way of saying this.

Did I say succinct? If by *succint* I mean *long and boring inductive definition* then yes, it is succint. All we really *want* to say is
$(\mathcal{L};\vec{a}) \models \phi(\vec{x})$ *iff the statement $\phi(\vec{a})$ is true in $\mathcal{L}$.*

Why don't we just say that? Because we need the inductive definition in order to prove things about $\models$. Alas, in order to prove things rigorously we must be a bit pedantic.

Definition 3.8 Let $\mathcal{L} = (L,<)$ and $\vec{a} \in L^k$. Let $\phi(\vec{x})$ have k free variables. We define $(\mathcal{L};\vec{a}) \models \phi(\vec{x})$ inductively.

1. $(\mathcal{L};a_1,a_2) \models (x<y)$ holds iff $a_1 < a_2$).

2. $(\mathcal{L};a_1,a_2) \models (x=y)$ holds iff $a_1 = a_2$).

3. $(\mathcal{L};\vec{a}) \models \neg\phi(\vec{x})$ holds iff it is NOT the case that $(\mathcal{L},\vec{a}) \models \phi(\vec{x})$.

4. $(\mathcal{L};\vec{a}) \models \phi_1(\vec{x}) \wedge \phi_2(\vec{x})$ holds iff $(\mathcal{L},\vec{a}) \models \phi_1(\vec{x})$ and $(\mathcal{L},\vec{a}) \models \phi_2(\vec{x})$.

5. $(\mathcal{L};\vec{a}) \models \phi_1(\vec{x}) \vee \phi_2(\vec{x})$ holds iff $(\mathcal{L},\vec{a}) \models \phi_1(\vec{x})$ or $(\mathcal{L},\vec{a}) \models \phi_2(\vec{x})$.

6. $(\mathcal{L},\vec{a}) \models (\exists x)[\phi(\vec{x},x)]$ holds iff there is an $a \in L$ such that $(\mathcal{L};\vec{a},a) \models \phi(\vec{a},a)$.

7. To summarize: If $\phi(\vec{x})$ has k free variables and $\vec{a} \in L^k$ then $(\mathcal{L};\vec{a}) \models \phi(\vec{x})$ means that $\phi(\vec{a})$ is true in $\mathcal{L}$.

Definition 3.9 Let $\vec{a} \in L_1^k$ and $\vec{b} \in L_2^k$. $(\mathcal{L}_1;\vec{a}) \equiv_m^T (\mathcal{L}_2;\vec{b})$ if, for all $\phi(\vec{x})$ with $\mathrm{qd}(\phi) \le m$ and k free variables,

$$(\mathcal{L}_1;\vec{a}) \models \phi(\vec{x}) \text{ iff } (\mathcal{L}_2;\vec{b}) \models \phi(\vec{x}).$$

Definition 3.10 Let $m \ge 1$. A formula is *m-simple* if it is of the form $(\exists x)[\psi(\vec{x},x)]$ where $\mathrm{qd}(\psi(\vec{x},x)) \le m-1$.

We leave the proof of the following easy lemma to the reader.

Lemma 3.11 *Let $m \ge 1$. If $\mathrm{qd}(\phi(\vec{x})) = m$ then $\phi(\vec{x})$ can be written as a boolean combination of m-simple formulas.*

The following is our main theorem.

Theorem 3.12 *Let $\mathcal{L}_1 = (L_1, <)$ and $\mathcal{L}_2 = (L_2, <)$ be two linear orderings. For all $m \in \mathsf{N}$, for all $k \in \mathsf{N}$, for all $\vec{a} \in L_1^k$ and $\vec{b} \in L_2^k$. The following are equivalent:*

1. $(\mathcal{L}_1; \vec{a}) \equiv_m^G (\mathcal{L}_2; \vec{b}))$.
2. $(\mathcal{L}_1; \vec{a}) \equiv_m^T (\mathcal{L}_2; \vec{b})$.

Proof: We prove this by induction on m.
Base Case: $m = 0$. Assume $\vec{a}$ and $\vec{b}$ are k-tuples. We prove two implications.
First Implication:
$(\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b}) \implies (\mathcal{L}_1; \vec{a}) \equiv_0^T (\mathcal{L}_2; \vec{b})$.
Assume $(\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b}))$. Then, for all $1 \le i, j \le k$

$$a_i < a_j \text{ iff } b_i < b_j.$$

Hence $(\mathcal{L}_1; \vec{a})$ and $(\mathcal{L}_2, \vec{b})$ have the same order type.
We need to show, for all $\phi(\vec{x})$ of quantifier depth 0 that have $|\vec{a}|$ free variables, $(\mathcal{L}_1; \vec{a}) \models \phi(\vec{x})$ iff $(\mathcal{L}_2; \vec{b}) \models \phi(\vec{x})$. This means that they agree on all formulas of quantifier depth 0. Formulas of quantifier depth 0 are Boolean combinations of atomic formulas. One can easily show but induction on formation that since $(\mathcal{L}_1; \vec{a})$ and $(\mathcal{L}_2, \vec{b})$ have the same order type they will agree on any boolean combination of atomic formulas on $|\vec{a}|$ free variables.
Second Implication:
$(\mathcal{L}_1; \vec{a}) \equiv_0^T (\mathcal{L}_2; \vec{b}) \implies (\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b})$.
Assume $(\mathcal{L}_1; \vec{a}) \equiv_0^T (\mathcal{L}_2; \vec{b})$. Then for all $1 \le i, j \le k$

$$(\mathcal{L}_1; \vec{a}) \models x_i < x_j \text{ iff } (\mathcal{L}_2; \vec{b}) \models x_i < x_j.$$

Hence $a_i < a_j$ iff $b_i < b_j$. Therefore $(\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b})$.
Induction Step:
Induction Hypothesis: The statement of the theorem. However, we state it with the parameters that we need: For all $\vec{a} \in L_1^k$, $a \in L_1$, $\vec{b} \in L_2^k$, $b \in L_2$ the following are equivalent:

- $(L_1; \vec{a}, a) \equiv_m^T (L_2; \vec{b}, b)$.
- $(L_1; \vec{a}, a) \equiv_m^G (L_2; \vec{b}, b)$.

Let $k \in \mathsf{N}$, $\vec{a} \in L_1^k$, $\vec{b} \in L_2^k$. We need to prove two implications.
First Implication:
$(L_1; \vec{a}) \equiv_{m+1}^G (L_2; \vec{b}) \implies (L_1; \vec{a}) \equiv_{m+1}^T (L_2; \vec{b})$.
Assume $(L_1; \vec{a}) \equiv_{m+1}^G (L_2; \vec{b})$. (We won't use this until later.) We need to show that $(L_1; \vec{a}) \equiv_{m+1}^T (L_2; \vec{b})$.
Now we need to show, for all formulas $\phi(\vec{x})$ with $|\vec{a}|$ free variables, of quantifier depth $\le m+1$, $(\mathcal{L}_1; \vec{a}) \models \phi(\vec{z})$ iff $(\mathcal{L}_2; \vec{b}) \models \phi(\vec{z})$. We prove this by induction on the formation of $\phi(\vec{x})$.
CASE 1: If $\phi(\vec{x})$ has quantifier depth $\le m$ then, by $(L_1; \vec{a}) \equiv_m^G (L_2; \vec{b}) \implies (L_1; \vec{a}) \equiv_m^T (L_2; \vec{b})$. (a weaker statement than we have) and the induction hypothesis we have $(\mathcal{L}_1; \vec{a}) \models \phi(\vec{z})$ iff $(\mathcal{L}_2; \vec{b}) \models \phi(\vec{z})$.

CASE 2: (This is the main case of interest.) $\phi(\vec{x})$ is m-simple. $\phi(\vec{x}) = (\exists x)[\psi(\vec{x}, x)]$.

$$(\mathcal{L}_1; \vec{a}) \models \phi(\vec{x})$$

iff

$$(\mathcal{L}_1; \vec{a}) \models (\exists x)[\psi(\vec{x}, x)]$$

iff

there is an $a \in L_1$ such that

$$(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x).$$

KEY: We call a *WITNESS TO THE TRUTH!* of $(\exists x)[\psi(\vec{x}, x)]$. We need to find an analogous witness in L_2. We use the game to find *WITNESS TO THE TRUTH!*. Duplicator's winning move is an analog of a and is used to locate the *WITNESS TO THE TRUTH!*.

Recall that $(\mathcal{L}_1; \vec{a}) \equiv^G_{m+1} (\mathcal{L}_2; \vec{b})$. Hence Duplicator has a winning strategy. If Spoiler were to make the move $a \in L_1$ as his first move then Duplicator has a response. Let b be that response. Note that by Fact 3.7 $(\mathcal{L}_1; \vec{a}, a) \equiv^G_{m+1-1} (\mathcal{L}_2; \vec{b}, b)$. By the induction hypothesis applied to $\psi(\vec{x}, x)$

$$(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x) \text{ iff } (\mathcal{L}_2; \vec{b}, b) \models \psi(\vec{x}, x).$$

Hence

$$(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x)$$

iff

$$(\mathcal{L}_2; \vec{b}, b) \models \psi(\vec{x}, x)$$

iff

$$(\mathcal{L}_2; \vec{b}) \models (\exists x)[\psi(\vec{x}, x)].$$

iff

$$(\mathcal{L}_2; \vec{b}) \models \phi(\vec{x}).$$

Therefore we have

$$(\mathcal{L}_1; \vec{a}) \models \phi(\vec{x}) \text{ iff } (\mathcal{L}_2; \vec{b}) \models \phi(\vec{x}).$$

CASE 3: $\phi(\vec{x})$ is a boolean combination of m-simple formulas. This is an easy induction of formation.

By Lemma 3.11 we have covered all of the cases.

Second Implication:

$(\mathcal{L}_1; \vec{a}) \equiv^T_{m+1} (\mathcal{L}_2; \vec{b}) \implies (\mathcal{L}_1; \vec{a}) \equiv^G_{m+1} (\mathcal{L}_2; \vec{b})$

We need to find a winning strategy for Duplicator.

Assume that Spoiler plays $a \in L_1$ (the case where he plays $b \in L_2$ is similar). Duplicator needs to find a $b \in L_2$ that is analogous to $a \in L_1$.

Let F be the set of all formulas $\psi(\vec{x}, x)$ such that

- $\mathrm{qd}(\psi) \le m$.
- $|\vec{x}| = |\vec{a}|$.
- $(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x)$.

KEY: There are only a finite number of formulas in F.

Note that

$$(\mathcal{L}_1; \vec{a}, a) \models \bigwedge_{\psi \in F} \psi(\vec{x}, x).$$

Hence

$$(\mathcal{L}_1; \vec{a}) \models (\exists x)[\bigwedge_{\psi \in F} \psi(\vec{x}, x)].$$

KEY: $\mathrm{qd}((\exists x)[\bigwedge_{\psi \in F} \psi(\vec{x}, x))] = m + 1$. Hence, since $(\mathcal{L}_1; \vec{a}) \equiv^T_{m+1} (L_2; \vec{b})$

$$(\mathcal{L}_2; \vec{b}) \models (\exists x)[\bigwedge_{\psi \in F} \psi(\vec{x}, x)].$$

Let b be the witness. Hence

$$(\mathcal{L}_2; \vec{b}, b) \models \bigwedge_{\psi \in F} \psi(\vec{x}, x)].$$

KEY: We use the logical equivalence to get the witness which will give us the winning move for Duplicator.

We claim that if Duplicator plays b then he can win the game. Note that we now need to show

$$(\mathcal{L}_1; \vec{a}, a) \equiv^G_m (\mathcal{L}_2; \vec{b}, b).$$

By the induction hypothesis it will suffice to show

$$(\mathcal{L}_1; \vec{a}, a) \equiv^T_m (\mathcal{L}_2; \vec{b}, b).$$

Let $\phi(\vec{x}, x)$ be a formula such that $\mathrm{qd}(\phi(\vec{x}, x)) = m$ and $|\vec{x}| = |\vec{a}|$. Assume

$$(\mathcal{L}_1; \vec{a}, a) \models \phi(\vec{x}, x).$$

Then $\phi \in F$.

Since

$$(\mathcal{L}_2; \vec{b}, b) \models \bigwedge_{\psi \in F} \psi(\vec{x}, x)].$$

We have that

$$(\mathcal{L}_2; \vec{b}, b) \models \phi(\vec{x}, x)].$$

$\blacksquare$

4 Genearlization

Section 2 was about the Duplicator-Spoiler Game on a linear order. However, this can be generalized.

Exercise 1 Define Duplicator-Spoiler games on pairs of graphs. State and prove an analog of Theorem 3.4.

Exercise 2 Define Duplicator-Spoiler games on pairs of hypergraphs. State and prove an analog of Theorem 3.4.

References

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[5] J. G. Rosenstein. *Linear Orderings.* Academic Press, New York, 1981. Pure and Applied Math Series 98.