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THE ERDŐS-SZEKERES PROBLEM ON POINTS IN CONVEX POSITION – A SURVEY

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Abstract. In 1935 Erdős and Szekeres proved that for any integer $n \geq 3$ there exists a smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane contains n points that are the vertices of a convex n-gon. They also posed the problem to determine the value of $N(n)$ and conjectured that $N(n) = 2^{n-2} + 1$ for all $n \geq 3$.

Despite the efforts of many mathematicians, the Erdős-Szekeres problem is still far from being solved. This paper surveys the known results and questions related to the Erdős-Szekeres problem in the plane and higher dimensions, as well as its generalizations for the cases of families of convex bodies and the abstract convexity setting.

1. Introduction

The following problem attracts the attention of many mathematicians by its beauty and elementary character.

The Erdős-Szekeres Problem 1.1. ([32], [33]) *For any integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (i.e., no three of the points are on a line) contains n points that are the vertices of a convex n-gon.*

The interest of Erdős and Szekeres in this problem was initiated by Esther Klein (later Mrs. Szekeres), who observed that any set of five points in general position in the plane contains four points that are the vertices of a convex quadrilateral. Indeed, there are three distinct types of placement of five points in the plane, no three on a line, as shown on Figure 1.1. In any of these cases, one can pick out at least one convex quadrilateral determined by the points.

Klein suggested the following more general problem, namely the problem on the existence of a finite number $N(n)$ such that from any set containing at least $N(n)$ points in general position in the plane, it is possible to select n points forming a convex polygon.

As observed by Erdős and Szekeres, there are two particular questions related to this problem:

(1) *Does the number $N(n)$ exist?*

(2) *If so, how is $N(n)$ determined as a function of n?*

In their paper [32], Erdős and Szekeres proved the existence of the number

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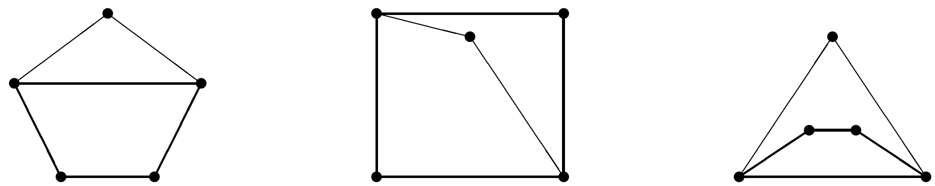

FIGURE 1.1. Any five points in general position determine a convex quadrilateral.

$N(n)$ by two different methods. The first of them uses Ramsey's theorem and, as a result, gives the inequality $N(n) \leq R_4(5, n)$, where $R_4(5, n)$ is a Ramsey number (see Section 2 for details). The second method is based on some geometric considerations, resulting in a better upper bound $N(n) \leq \binom{2n-4}{n-2} + 1$. In the same paper Erdős and Szekeres formulated the following conjecture.

Conjecture 1.2 ([32]). *$N(n) = 2^{n-2} + 1$ for all $n \geq 3$.*

Many years later (see [25], [26], and [29]) Erdős stated that "Szekeres conjectured $N(n) = 2^{n-2}+1$." He amended this in [28] to "Probably $N(n) = 2^{n-2}+1$." Szekeres was more forceful in [78], saying "Of course we firmly believe that $N(n) = 2^{n-2}+1$ is the correct value." Another statement of faith in the conjectured value for $N(n)$ may be found in [16]. Shortly before he died, Erdős [30] wrote: "I would certainly pay $500 for a proof of Szekeres' conjecture."

Klein and Szekeres married shortly after the publication of [32], prompting Erdős to call Problem 1.1 the "Happy End Problem". The books [45] and [73] contain picturesque descriptions of the Erdős-Szekeres problem origins.

Their second paper [33] contains an example of a set of $2^{n-2}, n \geq 3$, points in general position in the plane, no n of which determine a convex polygon. In other words, Erdős and Szekeres have shown that $N(n) \geq 2^{n-2} + 1$ for all $n \geq 3$.

Despite its elementary character and the effort of many mathematicians, the Erdős-Szekeres problem is solved for the values $n = 3, 4$, and 5 only. The case $n = 3$ is trivial, and $n = 4$ is due to Klein. The original paper by Erdős and Szekeres [32] notes that E. Makai proved the equality $N(5) = 9$, while the first published proof of this result is due to Kalbfleisch *et al.* [50].

The next step in solving Problem 1.1 is to answer the following question.

Question 1.3. *Does any set of at least 17 points in general position in the plane contain 6 points that are the vertices of a convex hexagon?*

For larger values of n, the best known upper bound $N(n) \leq \binom{2n-5}{n-3} + 2$ was recently proved by Tóth and Valtr [80].

Later Erdős posed a similar problem on empty convex polygons.

Problem 1.4 ([27]). *For any positive integer $n \geq 3$, determine the smallest positive integer $H(n)$, if it exists, such that any set X of at least $H(n)$ points in general position in the plane contains n points which are the vertices of an empty convex polygon, i.e., a polygon whose interior does not contain any point of X.*

Trivially, $H(3) = 3$ and $H(4) = 5$, as easily follows from Figure 1.1. In 1978 Harborth [43] proved that $H(5) = 10$, while Horton [46] showed in 1983 that $H(n)$

does not exist for all $n \geq 7$. The only open case in Problem 1.4 is given by the following question of Erdős.

Question 1.5 ([29]). *Does the number $H(6)$ exist?*

There were many attempts to prove or disprove the existence of $H(6)$ (see Section 3 for details). The best known result in this direction belongs to Overmars *et al.* [67], who showed in 1989 that $H(6) \geq 27$, if it exists.

To give positive or negative answers to Questions 1.3 and 1.5, several algorithms for detecting a largest convex polygon or a largest empty convex polygon determined by a given set of points were elaborated. Detailed descriptions of these algorithms can be found in [3], [18], [22], [67].

Due to a wide interest in Problem 1.1, the original papers by Erdős and Szekeres were reprinted (see [32], [33], and [78]), and a long list of reviews and books discussed the problem in broad strokes (see [17], [20], [21], [31], [35], [39], [40], [41], [42], [45], [56], [59], [61], [73], [74], [85]). Nevertheless, none of them covers the whole variety of existing results and open problems. The purpose of this survey is to reflect the recent stage of the Erdős-Szekeres problem, as well as its various generalizations and related questions. The content of the survey is indicated by section headings as follows.

1. Introduction.
2. Bounds on $N(n)$.
3. The Erdős problem on empty convex polygons.
4. Higher dimensional extensions.
5. Other generalizations and related results.

In particular, we formulate a new conjecture for the higher dimensional version of the Erdős-Szekeres problem (see Conjecture 4.2).

In what follows, we use standard notation: E^d denotes Euclidean d-space; $|X|$ and conv$X$ are the cardinality and the convex hull of a set $X \subset E^d$, respectively; ext$P$ is the set of vertices (extreme points) of a convex polytope $P \subset E^d$.

We say that a set $X \subset E^d$ is in *convex position* if $x \notin \text{conv}(X \setminus x)$ for any point $x \in X$. In other words, $X \subset E^d$ is in convex position provided X is a set of vertices of a convex polytope in E^d. A convex polytope whose vertices belong to a set $Y \subset E^d$ is called *empty* provided the interior of the polytope does not contain any point of Y. Recall that a set $Z \subset E^d$ is in *general position* if no $d+1$ of its points lie in a hyperplane.

2. Bounds on $N(n)$

2.1. **Upper Bounds from Ramsey Theory.** As was mentioned above, the first proof on the existence of $N(n)$ belongs to Erdős and Szekeres [32] and is based on the following fundamental result of Ramsey [72].

Theorem 2.1 ([72]). *For any positive integers $k, l_1, l_2, \dots, l_r$ there exists a smallest positive integer m_0 satisfying the following condition. For any integer $m \geq m_0$, if the k-element subsets of $\{1, 2, \dots, m\}$ are colored with colors $1, 2, \dots, r$, then there exists an $i, 1 \leq i \leq r$, and an l_i-element subset $T \subset \{1, 2, \dots, m\}$ so that each of the k-element subsets of T is i-colored.*

The smallest number m_0 for which the conclusion of Ramsey's theorem holds is usually denoted by $R_k(l_1, l_2, \dots, l_r)$. A proof of Ramsey's theorem, for the case

$r = 2$, was independently discovered by Szekeres (see [32]) for the purpose of showing the finiteness of $N(n)$.

The paper [38] discusses the importance of [32] in the development of Ramsey theory. In the following theorem we give three different methods for getting upper bounds on $N(n)$ based on Ramsey's theorem. In each case, an easy argument shows that $N(n) \leq R_k(l_1, l_2)$ for an appropriate choice of k, l_1, l_2.

Theorem 2.2. *For any positive integer $n \geq 3$ the number $N(n)$ exists and*

$$N(n) \leq \min\{R_4(n, 5), R_3(n, n)\}.$$

Proof. 1) Let X be any set of at least $R_4(n, 5)$ points in general position in the plane. The original proof of [32] colors the 4-element subsets of X with color 1 if the points are in convex position and colors them with color 2 otherwise. Klein's argument shows that it is impossible for all of the 4-element subsets of a 5-element subset of X to be of color 2. Hence it must be true that X contains an n-element subset for which all 4-element subsets are of color 1; i.e. all of them are in convex position. It easily follows that all the n points are in convex position. Hence $N(n) \leq R_4(n, 5)$.

2) Lewin [60] reported that Tarsy, an undergraduate student, had come up with the following independent proof while taking an exam in a combinatorics course. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of points in general position in the plane, with $m \geq R_3(n, n)$. Color a 3-element subset $\{x_i, x_j, x_k\} \subset X$, $i < j < k$, with color 1 if one encounters the points in the order (x_i, x_j, x_k) by passing clockwise around their convex hull. Color the subset with color 2 otherwise. It is easy to see that a 4-element subset of X is in convex position if and only if all of its 3-element subsets are colored with the same color. This implies that an n-element subset of X is in convex position if and only if all of its 3-element subsets are colored with the same color. Hence $N(n) \leq R_3(n, n)$.

3) The most recent proof involving a Ramsey-theoretic upper bound on $N(n)$ was discovered by Johnson [49]. Color a 3-element subset S of a planar set X in general position with color 1 if there is an even number of points of X in the interior of conv$S$, and color S with color 2 otherwise. A one-line proof then shows that a 4-element subset of X is in convex position if and only if all of its 3-element subsets have the same color. This once again implies that an n-element subset of X is in convex position if and only if all of its 3-element subsets have the same color. Therefore, $N(n) \leq R_3(n, n)$. $\square$

Lewin [60] points out that $R_3(n, n)$ seems to be lower than $R_4(n, 5)$. The best known bounds on the Ramsey numbers $R_3(n, n)$ are $2^{bn^2} \leq R_3(n, n) \leq 2^{c^n}$, for some constants b and c (see [39]). The next section shows that these bounds are far from the true value of $N(n)$.

2.2. **Caps and Cups - Better Upper Bounds.** We will assume in this section that a coordinate system (x, y) is introduced in the plane. Let $X = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$ be a set of points in general position in the plane, with $x_i \neq x_j$ for all $i \neq j$. A subset of points $\{(x_{i_1}, y_{i_1}), (x_{i_2}, y_{i_2}), \ldots, (x_{i_r}, y_{i_r})\}$ is called in [16] an *r-cup* if $x_{i_1} < x_{i_2} < \ldots < x_{i_r}$ and

$$\frac{y_{i_1} - y_{i_2}}{x_{i_1} - x_{i_2}} < \frac{y_{i_2} - y_{i_3}}{x_{i_2} - x_{i_3}} < \ldots < \frac{y_{i_{r-1}} - y_{i_r}}{x_{i_{r-1}} - x_{i_r}}.$$

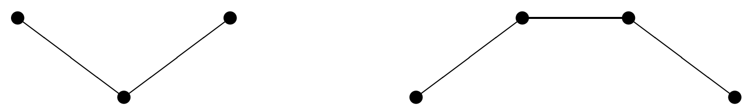

FIGURE 2.1. Examples of a 3-cup and a 4-cap.

Similarly, the subset is called an *r-cap* if $x_{i_1} < x_{i_2} < \ldots < x_{i_r}$ and

$$\frac{y_{i_1} - y_{i_2}}{x_{i_1} - x_{i_2}} > \frac{y_{i_2} - y_{i_3}}{x_{i_2} - x_{i_3}} > \ldots > \frac{y_{i_{r-1}} - y_{i_r}}{x_{i_{r-1}} - x_{i_r}}.$$

In other words, the set of r points forms an r-cup (respectively, an r-cap) provided the sequence of slopes of the segments

$$[(x_{i_1}, y_{i_1}), (x_{i_2}, y_{i_2})], [(x_{i_2}, y_{i_2}), (x_{i_3}, y_{i_3})], \ldots , [(x_{i_{r-1}}, y_{i_{r-1}}), (x_{i_r}, y_{i_r})]$$

is monotonically increasing (respectively, decreasing). See, e.g., Figure 2.1.

Define $f(k, l)$ to be the smallest positive integer for which X contains a k-cup or an l-cap whenever X has at least $f(k, l)$ points.

Theorem 2.3 ([32]). $f(k, l) \leq \binom{k+l-4}{k-2} + 1$.

Proof. The inequality follows from the boundary conditions $f(k, 3) = f(3, k) = k$ and the recurrence $f(k, l) \leq f(k-1, l) + f(k, l-1) - 1$. We sketch a proof of the recurrence.

Suppose that X contains $f(k-1, l) + f(k, l-1) - 1$ points. Let Y be the set of left endpoints of $(k-1)$-cups of X. If $X \setminus Y$ contains $f(k-1, l)$ points, then it contains an l-cap. Otherwise, Y contains $f(k, l-1)$ points. Suppose that Y contains an $(l-1)$-cap $\{(x_{i_1}, y_{i_1}), (x_{i_2}, y_{i_2}), \ldots , (x_{i_{l-1}}, y_{i_{l-1}})\}$. Let $\{(x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), \ldots , (x_{j_{k-1}}, y_{j_{k-1}})\}$ be a $(k-1)$-cup with $i_{l-1} = j_1$. A quick sketch then shows that either $(x_{i_{l-2}}, y_{i_{l-2}})$ can be added to the $(k-1)$-cup to create a k-cup or (x_{j_2}, y_{j_2}) can be added to the $(l-1)$-cap to create an l-cap. $\square$

Because $N(n) \leq f(n, n)$, we get from Theorem 2.3 that $N(n) \leq \binom{2n-4}{n-2} + 1$.

This upper bound was not improved upon until 63 years later, when Chung and Graham [16] managed to modify the above argument to show that $N(n) \leq \binom{2n-4}{n-2}$. A further modification by Kleitman and Pachter [57] implied $N(n) \leq \binom{2n-4}{n-2} + 7 - 2n$. Shortly thereafter, Tóth and Valtr [80] gave the following simple argument to roughly cut the Erdős-Szekeres bound in half.

Theorem 2.4 ([80]). $N(n) \leq \binom{2n-5}{n-3} + 2$.

Proof. Let a be an extreme point of a planar set X. Denote by b a point outside of conv$X$ so that no line determined by the points of $X \setminus \{a\}$ intersects the segment $[a, b]$. Let l be a line through b that does not intersect conv$X$. It is easily seen that the projective transformation T that maps l to the line at infinity and maps the segment $[a, b]$ to the vertical ray emanating downward from $T(a)$ has the following properties:

(i) a subset Y of X is in convex position if and only if $T(Y)$ is,

(ii) a subset Z of X containing a is in convex position if and only if $T(Z \setminus a)$ is a cap.

Suppose now that X contains $\binom{2n-5}{n-3}+2$ points. Then $T(X \setminus a)$ determines either a $(n-1)$-cap or a n-cup (see Theorem 2.3). This implies that X contains n points in convex position. $\square$

Stirling's formula shows that $\binom{2n-5}{n-3}$ is smaller than 4^n and is, asymptotically, larger than $(4-c)^n$ for any constant $c > 0$. Chung and Graham [16] have offered \$100 for the first proof that $N(n) = O((4-c)^n)$ for some constant $c > 0$. (They offer no money for showing that no such c exists.) The simplicity of the argument of Tóth and Valtr [80] seems to indicate that further reductions in the upper bound are within reach. On the other hand, substantially different techniques might be needed to claim the \$100 prize.

2.3. **Construction for the Lower Bound.** We begin with a theorem that states that the inequality for $f(k,l)$ in Theorem 2.3 is actually an equality.

Theorem 2.5 ([32]). $f(k,l) = \binom{k+l-4}{k-2} + 1$.

Proof. Note that we have already observed this to be the case when k or l is 3. We proceed by induction. Suppose that we have a set A of $\binom{k+l-5}{k-3}$ points with no $(k-1)$-cup and no l-cap, and a set B of $\binom{k+l-5}{k-2}$ points with no k-cup and no $(l-1)$-cap. Translate these sets so that the following conditions are satisfied:

(i) every point of B has greater first coordinate than the first coordinates of points of A,

(ii) the slope of any line connecting a point of A to a point of B is greater than the slope of any line connecting two points of A or two points of B.

Let $X = A \cup B$ be the resulting configuration. Any cup in X that contains elements of both A and B may have only one element of B. It follows that X contains no k-cup. We similarly see that X contains no l-cap. Thus

$$f(k,l) \geq \binom{k+l-5}{k-3} + \binom{k+l-5}{k-2} + 1 = \binom{k+l-4}{k-2} + 1. \qquad \square$$

Now we are ready to prove the inequality $N(n) \geq 2^{n-2}+1$. This lower bound on $N(n)$ was essentially proved in [33]. Some inaccuracies in the proof were later corrected by Kalbfleisch and Stanton [51]. (Erdős refers to their corrections in [26].) We sketch the main ideas of the construction as it is presented in [61].

Theorem 2.6 ([33],[51])). $N(n) \geq 2^{n-2}+1$.

Proof. To prove the inequality, we construct a set X of 2^{n-2} points with no subset of n points in convex position. For $i = 0, 1, \ldots, n-2$, let T_i be a set of $\binom{n-2}{i}$ points containing no $(i+2)$-cap and no $(n-i)$-cup and having the property that no two points in the set are connected by a line having slope of absolute value greater than 1.

For $i = 0, 1, \ldots, n-2$, place a small copy of T_i in a neighborhood of the point on the unit circle making an angle of $\frac{\pi}{4} - \frac{i\pi}{2(n-2)}$ with the positive x-axis. Let X be the union of the T_i, $i = 1, 2, \ldots, n-2$. Then

$$|X| = \sum_{i=0}^{n-2} \binom{n-2}{i} = 2^{n-2}.$$

Suppose that Y is a subset of X in convex position. Let k and l be the smallest and the largest values of i so that $Y \cap T_i \neq \emptyset$. If $k = l$, then Y contains no $(k+2)$-cap and no $(n-k)$-cup. The construction guarantees that:

(a) $Y \cap T_k$ is a cap of at most $k+1$ points,
(b) $Y \cap T_l$ is a cup of at most $n-l-1$ points,
(c) $|Y \cap T_i| \leq 1$ for all $i = k+1, k+2, \dots, l-1$.

Thus

$$|Y| \leq k + 1 + l - k - 1 + n - l - 1 = n - 1.$$

Hence no subset of X in convex position contains n points. □

An interesting conjecture was formulated by Erdős *et al.* [34] that connects the proof of the upper bound on $N(n)$ given in [32] to the conjectured lower bound on $N(n)$. Let $m(n,k,l)$ be the smallest number such that any set of $m(n,k,l)$ points in general position in the plane contains either a set of n points in convex position, or a k-cup, or an l-cap. It is proved in [34] that $m(n,k,l) \leq \sum_{i=n-l}^{k-2} \binom{k-2}{i}$. The authors of [34] conjecture that equality holds and prove its equivalence to the conjecture $N(n) = 2^{n-2} + 1$.

2.4. **The case** $n = 5$**.** Erdős and Szekeres note already in [32] that Makai had proved the equality $N(5) = 9$. Credit for this result is given in [33] to Makai and Turán. A proof did not appear in the literature until Kalbfleisch *et al.* [50]. Since their proof was rather long, Bonnice [12] and Lovász ([61], pp. 88–89, 501–506) independently published much simpler proofs. In what follows we outline the proof of Bonnice [12].

Theorem 2.7. $N(5) = 9$.

The proof of Theorem 2.7 is based on Lemma 2.8 below. Given a finite set X of points in the plane, the statement that X is $(k_1, k_2, \dots, k_j)$ will mean that $|X| = k_1 + k_2 + \dots + k_j$ and the convex hull of X is a k_1-gon; that, when the vertex set of conv$X$ is taken away from X, the convex hull of the remaining points is a k_2-gon; *etc.* Also, if *abcd* is a convex quadrilateral with vertices ordered counterclockwise, *beam ab*:*cd* denotes the section of the plane obtained by deleting conv$\{a,b,c,d\}$ from the convex section of the plane bounded by segment $[a,b]$ and rays $[a,d)$, $[b,c)$. Similarly, if x, y, and z are not collinear, *beam* x:yz will denote the infinite section of the plane obtained by deleting conv$\{x,y,z\}$ from the convex cone which has vertex x and is bounded by rays $[x,y)$ and $[x,z)$.

Lemma 2.8. *If a planar set* Y *is* $(3,3,2)$*, or* $(4,3,1)$*, or* $(3,4,2)$*, then* Y *determines a convex pentagon.*

Proof. First, assume that Y is $(3,3,2)$. Let y_1, y_2, y_3 be the vertices of conv$Y$; let triangle $v_1v_2v_3$ be the second triangle, conv$(Y \setminus \{y_1,y_2,y_3\})$; and let z_1, z_2 be the two points of Y interior to $v_1v_2v_3$. We may assume that line (z_1, z_2) intersects sides $[v_1,v_2]$ and $[v_1v_3]$ of $v_1v_2v_3$ such that $[z_1,z_2)$ intersects $[v_1,v_2)$.

The vertices y_1, y_2, and y_3 of the outside triangle are in the union of beams z_1z_2:v_2v_3, z_1:v_1v_3, and z_2:v_1v_2. If one of these vertices, say y_1, is in beam z_1z_2:v_2v_3, then $z_1z_2v_2y_1v_3$ is a convex pentagon. Thus we may assume that two of the points y_1, y_2, y_3, say y_1 and y_2, are in beam z_1:v_1v_3. Since conv$\{y_1,y_2,y_3\}$ contains all of v_1, v_2, v_3, triangle $v_1v_2v_3$ lies in one of the open half-planes determined

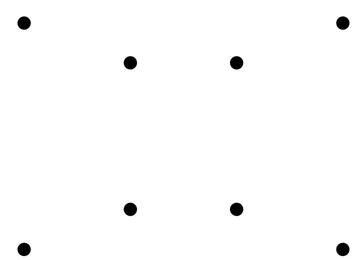

Figure 2.2. A set of eight points determining no convex pentagon.

by line (y_1, y_2). Thus line (y_1, y_2) does not intersect $\text{conv}\{v_1, v_2, v_3\}$, and therefore $z_1 v_1 v_3 y_1 y_2$ is a convex pentagon.

Now assume that Y is $(4, 3, 1)$. Let y_1, y_2, y_3, and y_4 be the vertices of $\text{conv}Y$; let $v_1 v_2 v_3$ be the inside triangle, $\text{conv}(Y \setminus \{y_1, y_2, y_3, y_4\})$; and let z denote the point of Y inside it. Partitioning the plane outside $v_1 v_2 v_3$ into beams $z{:}v_1 v_3$, $z{:}v_1 v_2$, and $z{:}v_2 v_3$, we may assume that two of the four points y_1, y_2, y_3, y_4, say y_1 and y_2, are in beam $z{:}v_1 v_3$. Then, as above, $z v_1 v_3 y_1 y_2$ is a convex pentagon.

Finally, assume that Y is $(3, 4, 2)$. The technique is the same: let $y_1 y_2 y_3$, $v_1 v_2 v_3 v_4$, and $z_1 z_2$ be the triangle, quadrilateral, and line segment given by the fact that Y is $(3, 4, 2)$. If line (z_1, z_2) cuts a vertex, say v_1, of $v_1 v_2 v_3 v_4$, then $z_1 z_2 v_2 v_3 v_4$ is a convex pentagon. So assume that line (z_1, z_2) cuts sides $[v_1, v_4]$ and $[v_2, v_3]$ such that rays $[z_1, z_2)$ and $[v_2, v_3)$ intersect. As above, if there is a point of $\{y_1, y_2, y_3\}$ in one of the beams $z_1 z_2{:}v_3 v_4$, $z_2 z_1{:}v_1 v_2$, a convex pentagon is formed. Thus we may assume that $\{y_1, y_2, y_3\}$ lies in the union of beams $z_1{:}v_1 v_4$ and $z_2{:}v_2 v_3$. In particular, we may assume that both y_1 and y_2 are in $z_1{:}v_1 v_4$; whence, as before, $z_1 v_1 v_4 y_1 y_2$ is a convex pentagon. □

Proof of Theorem 2.7. If a set of nine points in general position in the plane determines no convex pentagon, it is one of the following: $(4, 4, 1)$, $(4, 3, 2)$, $(3, 4, 2)$, or $(3, 3, 3)$. Each of the first two cases has a subset of 8 points which is $(4, 3, 1)$, and each of the last two cases has a subset which is $(3, 3, 2)$. Thus Lemma 2.8 applies to all cases to show that $N(5) \leq 9$. The opposite inequality easily follows from Figure 2.2. □

As mentioned by Bonnice [12], the same approach can hardly be applied to the case $n = 6$. Indeed, assuming that $N(6) = 17$, one can see that a set X of 17 points in the plane can determine 70 distinct tuples $(k_1, k_2, \ldots, k_j)$ representing the different ways the successive convex hulls of X might nest if it determines no convex hexagon.

3. The Erdős problem on empty polygons

In 1978 Erdős [27], [28], [29] posed a new problem on convex polygons.

Problem 3.1 ([27])**.** *For any positive integer $n \geq 3$, determine the smallest positive integer $H(n)$, if it exists, such that any set X of at least $H(n)$ points in general position in the plane contains n points which are the vertices of an empty convex polygon, i.e., a polygon whose interior does not contain any point of X.*

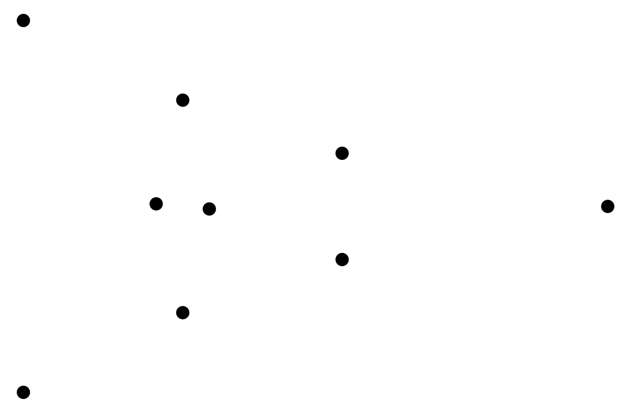

FIGURE 3.1. A set of nine points with no empty convex pentagon.

Trivially, $H(3) = 3$, and from Figure 1.1 we easily conclude that $H(4) = 5$. Using a direct geometric approach, Harborth [43] showed in 1978 that $H(5) = 10$. The inequality $H(5) \geq 10$ immediately follows from Figure 3.1, where a set of nine points in general position determines no empty convex pentagon (there are still two convex pentagons, neither being empty).

In 1983 Horton [46] showed that $H(n)$ does not exist for all $n \geq 7$. This statement is due to the following analytic construction of a planar set S_k of 2^k $(k \geq 1)$ points in general position determining no empty convex 7-gon. Let $a_1 a_2 \cdots a_k$ be the binary representation of the integer i, $0 \leq i < 2^k$, where leading 0's are omitted. Put $c = 2^k + 1$ and define $d(i) = \sum_{j=0}^{k} a_j c^{j-1}$. Now a simple analytical consideration shows that any convex polygon determined by the set $S_k = \{(i, d(i)) : i = 0, 1, \ldots, 2^k - 1\}$ has at most six vertices.

Valtr [82] defines a *Horton set* inductively as follows. The empty set and any one-point set are Horton sets. The points of a Horton set H are in general position in the plane, with distinct x-coordinates. Furthermore, H can be partitioned into two sets A and B such that:

1. Each of A and B is a Horton set.
2. The set A is below any line connecting two points of B, and the set B lies above any line connecting two points of A.
3. The x-coordinates of the points of A and B alternate.

One can easily prove by induction on $n = |H|$ that if $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ is a 4-cup (respectively, 4-cap) in H, then there is a point (x, y) of H lying above (respectively, below) one of the segments $[(x_i, y_i), (x_{i+1}, y_{i+1})]$, $i = 1, 2, 3$. It immediately follows that H contains no empty 7-gon, because otherwise A would contain a 4-cup or B would contain a 4-cap.

Note that the above sets S_k fit the definition of Horton sets. Valtr [82], [83] uses Horton sets in several generalizations of the empty polygon problem, as we will see in Sections 4 and 5.

In this connection the following question of Erdős [29] (and later of Horton [46]) still remains open.

Question 3.2. *Does the number $H(6)$ exist?*

Horton [46] expresses the belief that $H(6)$ exists. Bárány and Valtr [86] present a conjecture which would imply the existence of $H(6)$. Trying to determine the lower bound on $H(6)$, Avis and Rappaport [3] elaborated a method to determine whether a given set of points in the plane contains an empty convex 6-gon, and by using this approach they found a set of 20 points in general position containing no empty convex 6-gon.

Overmars *et al.* [67], modifying considerations of Dobkin *et al.* [22], constructed an algorithm of time complexity $O(n^2)$ that solves the following problem: for a given set V in the plane, containing no empty convex 6-gon, and for a point $z \notin V$, determine whether the set $\{z\} \cup V$ contains an empty convex 6-gon. Using this algorithm, they found a set of 26 points containing no empty convex 6-gon. Hence, $H(6) \geq 27$, if it exists.

As with the original Erdős-Szekeres problem, the theory for the empty polygon problem is limited to that which can be proved using cups and caps. There remains once again a large gap that probably will require some new paradigms to be bridged.

4. Higher dimensional extensions

4.1. **The Erdős-Szekeres Problem in Higher Dimensions.** An observation that the Erdős-Szekeres problem can be generalized for higher dimensions was already mentioned by its authors (see [32]) and later rediscovered by Danzer *et al.* [21]. Recall that a set X of points in Euclidean space E^d is in *general position* if no $d+1$ points of X lie in a hyperplane. (Clearly, a set X is in general position if and only if for any positive integer k, $1 \leq k \leq d$, no $k+1$ points of X lie in a k-dimensional plane.) Furthermore, X is said to be in *convex position* if no point of X lies in the convex hull of the remaining points. In other words, a set X in general position is in convex position if and only if it is the vertex set of a convex d-polytope in E^d.

Following [21], we define $N_d(n)$, $d \geq 2, n \geq 1$ to be the smallest positive integer such that any set of $N_d(n)$ points in general position in E^d contains n points in convex position. Similarly to the planar case, one can pose the following two problems:

1) *Do the numbers $N_d(n)$ exist for all $d \geq 2$ and $n \geq 1$?*

2) *If yes, what are the values of $N_d(n)$?*

The existence of $N_d(n)$ can be established analogously to the planar case (see Section 1) by implementing the following steps:

(a) A set X of at least $d+2$ points in general position in E^d is in convex position if and only if any $d+2$ of them are in convex position. (This fact is a direct consequence of Carathéodory's theorem [14]: a point z belongs to the convex hull of a set $A \subset E^d$ if and only if z belongs to the convex hull of at most $d+1$ points of A.)

(b) Any set of $d+3$ points in general position in E^d contains $d+2$ points in convex position. (A stronger version of this statement was proved by Motzkin [63]; see also [64], who showed that the number of nonconvex $(d+2)$-subsets of a general $(d+3)$-set in E^d equals either 0, or 2, or 4 for all $d \geq 2$.)

(c) If $R_{d+2}(n, d+3)$ is a Ramsey number, then any set of $R_{d+2}(n, d+3)$ points in general position in E^d contains n points in convex position. (This statement is a direct generalization of the original proof by Erdős and Szekeres for the planar case.)

As a result, we conclude that the numbers $N_d(n)$ exist for all positive integers $d \geq 2$ and $n \geq 1$, and $N_d(n) \leq R_{d+2}(n, d+3)$.

Valtr [85] gave another idea for proving the existence of numbers $N_d(n)$. He considers any set X of at least $N_2(n)$ points in general position in E^d and its projection Y onto a two-dimensional subspace $L \subset E^d$ such that Y is in general position in L. Since $|Y| \geq N_2(n)$, one can select in Y a subset of n points in convex position. It is easily seen that the prototypes of these points in X are in convex position. This consideration implies the inequality $N_d(n) \leq N_2(n)$, $d \geq 2$. A similar consideration is true for the case of an m-dimensional plane in E^d, $2 < m < d$. Hence we obtain

$$N_d(n) \leq N_{d-1}(n) \leq \ldots \leq N_2(n) \leq \binom{2n-5}{n-3} + 2.$$

Károlyi [52] has recently proved that $N_d(n) \leq N_{d-1}(n-1) + 1$, and this implies

$$N_d(n) \leq \binom{2n-2d-1}{n-d} + d.$$

The paper [52] also contains the intriguing result that for any $n \geq 1$ and $d \geq 3$ there is a smallest integer $M_d(n)$ so that if P is any set of $M_d(n)$ points in general position in E^d and if $p \in P$, then there is a subset of P consisting of n points in convex position and containing p.

Johnson [49] showed that his proof of the existence of $N(n)$ can be modified to get $N_d(n) \leq R_{d+1}(n, n, \ldots, n)$, where the last term has $d-1$ copies of n. We note here that no one has yet succeeded in generalizing the "caps and cups" arguments of Erdős and Szekeres [32] for the case $d \geq 3$.

The only known general lower bound for $N_d(n)$ is due to Károlyi and Valtr [54]. They prove that for each $d \geq 2$ there exists a constant $c = c(d)$ so that

$$N_d(n) = \Omega(c^{\sqrt[d-1]{n}}).$$

Table 4.1 shows the known values of $N_d(n)$, where the respective value is placed at the intersection of column d and row n.

Indeed, the equalities $N_d(n) = n$ if $n \leq d+1$ are trivial, and $N_d(d+2) = d+3$, mentioned by Danzer *et al.* [21] (see also Grünbaum [40]), was proved by Motzkin [63]. We note here that the equality $N_2(4) = 5$, which is due to Klein, is a particular case of $N_d(d+2) = d+3$. The next range of values of $N_d(n)$ is given by the following new theorem.

Theorem 4.1. $N_d(n) = 2n - d - 1$ *for* $d + 2 \leq n \leq \lfloor 3d/2 \rfloor + 1$.

Proof. As a consequence of a stronger assertion by Bisztriczky and Soltan [11] (see Section 4.2 below) one has $N_d(n) \leq 2n - d - 1$ for $d \geq 2$ and $d + 2 \leq n \leq \lfloor 3d/2 \rfloor + 1$. For any $n \geq d + 2$, Bisztriczky and Harborth [10] constructed a set $X = \{x_1, x_2, \ldots, x_{2n-d-3}\}$ in E^d so that $X \cup \{o\}$ is in general position and every set of $n-1$ points of X contains the origin o of E^d in the interior of its convex hull. They showed that no set of n points of $X \cup \{o\}$ could be the set of vertices of an empty convex polytope.

If we now scale the points of X by positive scalars λ_i so that for each $i \geq n$, $\lambda_i x_i$ is in the interior of the convex hull of every set of $n-1$ points of $\{\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_{i-1} x_{i-1}\}$, one can similarly see that no set of n points of $\{\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_{2n-d-3} x_{2n-d-3}, o\}$ is in convex position. Thus $N_d(n) \geq 2n-d-1$ for $n \geq d+2$. ☐

n \ d	2	3	4	5	6	7	8	9	…
14								18	
13							17	16	
12							15	14	
11						14	13	12	
10					13	12	11	10	
9					11	10	9	9	
8				10	9	8	8	8	
7			9	8	7	7	7	7	
6		9	7	6	6	6	6	6	
5	9	6	5	5	5	5	5	5	
4	5	4	4	4	4	4	4	4	
3	3	3	3	3	3	3	3	3	
2	2	2	2	2	2	2	2	2	
1	1	1	1	1	1	1	1	1	

TABLE 4.1. Known values of $N_d(n)$ when $d < 10$.

For $n > \lfloor 3d/2 \rfloor + 1$, there are known only two values of $N_d(n)$: $N_2(5) = 9$; see Section 2 and $N_3(6) = 9$, due to Bisztriczky and Soltan [11]. Their proof is similar to that of Bonnice and is based on selecting three subsets, P, Q, R, of a set $X \subset E^3$ of 9 points in general position such that

$$P = \operatorname{ext} \operatorname{conv} X, \quad Q = \operatorname{ext} \operatorname{conv}(X \setminus P), \quad R = \operatorname{ext} \operatorname{conv}(X \setminus (P \cup Q)).$$

Considering separately the cases

(i) $|P| \geq 6$,

(ii) $|P| = 5$ and $|Q| = 4$,

(iii) $|P| = 4$ and $|Q| = 5$,

(iv) $|P| = 4$, $|Q| = 4$, and $|R| = 1$,

they show each time the existence of a subset of 6 points in X in convex position.

It is interesting to mention that all the known values of $N_d(n)$, with $d \geq 2$ and $n > \lfloor (3d+1)/2 \rfloor$, satisfy the equality $N_d(n) = 4N_d(n-d) - 3$. This also agrees with the conjecture $N_2(n) = 2^{n-2} + 1$ of Erdős and Szekeres. We therefore offer the following conjecture.

Conjecture 4.2. $N_d(n) = 4N_d(n-d) - 3$ *for all* $d \geq 2$ *and* $n > \lfloor (3d+1)/2 \rfloor$.

Grünbaum ([40], pp. 22-23) discovered a variant of the Erdős-Szekeres problem in higher dimensions. Namely, he established the existence of the minimum number $B_d(n)$, $d \geq 2$ and $n \geq 1$, such that any set $X \subset E^d$ consisting of at least $B_d(n)$ points in general position contains a subset of n points lying on the boundary of a convex body in E^d. His proof is based on Ramsey's theorem and the following assertion: a finite set $Y \subset E^d$ lies on the boundary of a convex body in E^d if and only if each of its subsets of at most $2d+1$ points lies on the boundary of a convex body in E^d. The last statement is a direct consequence of the Steinitz theorem [77]: a point $z \in E^d$ belongs to the interior of the convex hull of a set $S \subset E^d$ if and only if z belongs to the interior of the convex hull of at most $2d$ points of S.

Later Bisztriczky and Soltan [11] showed that in the definition of $B_d(n)$ the set $X \subset E^d$ can be arbitrary (not necessarily in general position), and they also proved the equality $B_d(n) = N_d(n)$ for all $d \geq 2$ and $n \geq 1$. Their proof is based on a simple idea that any finite set in E^d can be approximated by a set in general position.

4.2. **Empty Convex Polytopes.** Generalizing the Erdős problem on empty convex polygons (see Section 2), Bisztriczky and Soltan [11] defined $H_d(n)$ to be the smallest positive integer, if it exists, such that any set X of $H_d(n)$ points in general position in E^d contains a subset of n points that are the vertices of an empty convex polytope, i.e., a polytope whose interior does not contain any point of X.

Valtr [83] proved the following deep results on the existence of $H_d(n)$, generalizing considerations of Horton [46].

1) $H_d(n)$ exists for all $n \leq 2d+1$, $d \geq 2$, and $H_d(2d+1) \leq N_d(4d+1)$;

2) $H_3(n)$ does not exist for all $n \geq 22$;

3) $H_d(n)$ does not exist if $n \geq 2^{d-1}(P(d-1)+1)$ and $d \geq 4$, where $P(d-1)$ is the product of the first $d-1$ prime numbers.

By using simple geometric arguments, Bisztriczky and Soltan [11] showed that $H_d(n) \leq 2n-d-1$ for $d+2 \leq n \leq \lfloor 3d/2 \rfloor + 1$. Later Bisztriczky and Harborth [10] proved the opposite inequality $H_d(n) \geq 2n-d-1$ if $H_d(n)$ exists. Their proof is based on the construction of a set $X \subset E^d$ of cardinality $2n-d-3$ such that the intersection of all convex hulls of subsets $Y \subset X$, $|Y| = n-1$, is nonempty. Combining the results of [10], [11] and Theorem 4.1, one gets

$$H_d(n) = N_d(n) = 2n-d-1 \quad \text{for} \quad d \geq 2 \quad \text{and} \quad d+2 \leq n \leq \lfloor 3d/2 \rfloor + 1. \tag{1}$$

For $n > \lfloor 3d/2 \rfloor + 1$, there are known only two values of $H_d(n)$: $H_2(5) = 10$, proved by Harborth [43], and $H_3(6) = 9$, proved by Bisztriczky and Soltan [11].

Since $H_3(6)$ can be considered as a particular case of $H_d(\lfloor (3d+1)/2 \rfloor + 1)$, we pose the following problem.

Problem 4.3. *Determine the value of* $H_d(\lfloor (3d+1)/2 \rfloor + 1)$.

Due to equality (1), it is sufficient to consider in the problem above the case when d is odd: e.g. the numbers $H_5(9)$, $H_7(12)$, $H_9(15)$, *etc.*

Some problems on the existence of empty convex polytopes in a two-colored set of points in E^d are considered by Borwein [13].

5. Other generalizations and related results

5.1. **Many Convex n-gons.** After the existence of convex n-gons has been proved, it is natural to ask how many there are. For a planar point set of r points in general position there are, of course, $\binom{r}{3}$ triangles determined by the set. The number of convex quadrilaterals formed by such a set is positive, for $r \geq 5$, as noted by Klein [32]. To show that the number of such convex quadrilaterals is at least $\binom{r-2}{3}$ was a problem in the Eleventh International Mathematical Olympiad, Bucharest, 1969 (see [88], [89], and [12]).

More generally, we can pose the following problem.

Problem 5.1. *Determine the minimum number of convex n-gons in a planar set of r points in general position.*

Clearly, a similar problem can be posed for higher dimensions.

Bárány and Valtr [5] proved that sufficiently large planar point sets have many collections of subsets in convex position of a given size n.

Theorem 5.2 ([5]). *For every integer $n \geq 4$ there is a constant $c_n > 0$ with the property that every sufficiently large finite planar set X in general position contains n subsets $Y_1, \ldots, Y_n$ with $|Y_i| \geq c_n|X|, i = 1, \ldots, n$, such that any set $\{y_1, \ldots, y_n\}$ satisfying $y_i \in Y_i$ for all $i = 1, \ldots, n$ is in convex position.*

Special cases of this theorem were previously proved by Solymosi [76] and Nielsen [66]. The infimum of the constants c_n for which Theorem 5.2 is true is shown in [5] to be at least

$$\left(N(n)2^{\binom{N(n)-1}{2}}\right)^{-1}.$$

A note at the end of [5] states that Solymosi has proved the inequality $c_n \geq 2^{-16n^2}$. Also proved in [5] is $c_4 \geq \frac{1}{22}$.

Erdős notes in [27] that a discussion with P. Hammer prompted him to study the function $s(r)$, the minimum number of convex subsets (of any number $n \geq 3$ of points) contained in a set of r points in general position in the plane. Erdős proves [27] that there exist constants a and b so that $r^{a\log r} < s(r) < r^{b\log r}$. He also speculates that $\lim_{n\to\infty} \log s(r)/(\log n)^2$ exists.

The problem of counting the number of empty n-gons has received considerable interest. Let $f_n(r)$ denote the minimum number of empty n-gons in a set of r points in general position in the plane. Note that Horton [46] proved $f_n(r) = 0$ for $n \geq 7$. Purdy [71] announced the equality $f_3(r) = O(r^2)$, while Harborth [43] showed that $f_3(r) = r^2 - 5r + 7$ for $3 \leq r \leq 9$ and $f_3(10) = 58$.

Katchalski and Meir [55] continued the investigation of $f_n(r)$ for smaller n by proving that $\binom{n}{2} \leq f_3(r) \leq Kr^2$ for some constant $K < 200$. Bárány and Füredi [4] followed up on the work of Katchalski and Meir by proving some new bounds:

$$r^2 - O(r\log r) \leq f_3(r) \leq 2r^2, \quad \frac{1}{4}r^2 - O(r) \leq f_4(r) \leq 3r^2,$$

$$\lfloor \frac{r}{10} \rfloor \leq f_5(r) \leq 2r^2, \quad f_6(r) \leq \frac{1}{2}r^2.$$

The upper bounds were improved by Valtr [84], and later by Dumitrescu [23], to

$$f_3(r) < 1.68r^2, \quad f_4(r) < 2.132r^2, \quad f_5(r) < 1.229r^2, \quad f_6 < 0.298r^2.$$

Valtr [84] mentioned personal correspondence from Bárány in which a lower bound of $f_4(r) \geq \frac{1}{2}r^2 - O(r)$ is given.

We close this section by mentioning the papers by Ambarcumjan [2], Karolyi [52], Hosono and Urabe [47], and Urabe [81], which deal with some combinatorial problems on clustering of finite planar sets, i.e. partitioning a set into subsets in convex position.

5.2. **Replacing Points with Convex Bodies.** Bisztriczky and Fejes Tóth (see [7], [8], [9]) showed that the Erdős-Szekeres problem has a generalization for the case of convex bodies in the plane. We say that a family of pairwise disjoint convex bodies is in *convex position* if none of its members is contained in the convex hull of the union of the others.

Theorem 5.3 ([7]). *For any integer $n \geq 4$, there is a smallest positive integer $g(n)$ such that if $\mathcal{F}$ is a family of pairwise disjoint convex bodies in the plane, any three of which are in convex position and $|\mathcal{F}| > g(n)$, then some n convex bodies of $\mathcal{F}$ are in convex position.*

The authors prove the existence of $g(n)$ using Ramsey's theorem. They also made the bold conjecture that $g(n) = N(n) - 1$. This conjecture is supported in [8], where it is shown that $g(5) = 8$.

A family $\mathcal{F}$ of convex bodies in the plane is said to have property H_k^n, for $3 \leq k < n$, if every set of k elements of $\mathcal{F}$ is in convex position and no n of them are in convex position. The third paper [9] of these authors is devoted to studying $h(k,n)$, which is the maximum cardinality of a family of mutually disjoint convex bodies satisfying property H_k^n. Clearly, $h(k,n) \leq h(3,n) = g(n)$, so $h(k,n)$ exists for all $3 \leq k < n$. An upper bound $h(4,n) \leq (n-4)\binom{2n-4}{n-2} - n + 7$, similar to the bound of [32] on $N(n)$, is proved. This is shown to imply the (very large) upper bound

$$g(n) \leq R_4((n-4)\binom{2n-4}{n-2} - n + 8, 5).$$

Smaller upper bounds are derived in [9] for $h(k,n)$ if $k \geq 5$.

The bounds on $h(k,n)$ from [9] are considerably improved by Pach and Tóth [69], [79]. The best known bounds are

$$2^{n-2} \leq h(3,n) \leq \binom{2n-4}{n-2}^2, \quad 2\lfloor \frac{n+1}{4} \rfloor^2 \leq h(4,n) \leq n^3,$$

$$n - 1 + \lfloor \frac{n-1}{3} \rfloor \leq h(5,n) \leq 6n - 12,$$
$$n - 1 + \lfloor \frac{n-1}{k-2} \rfloor \leq h(k,n) \leq \frac{k-4}{k-5}n, k > 5.$$

Pach and Solymosi [68] extend the results of Bárány and Valtr (see Section 5.1), replacing points by compact convex sets. Specifically, they prove that for every $n \geq 4$ there is a positive constant $c_n = 2^{O(n^2)}$, so that the following is true: every family $\mathcal{F}$ of r pairwise disjoint compact convex sets in general position in the plane has n disjoint $\lfloor c_n r \rfloor$-membered subfamilies $\mathcal{F}_i$, $1 \leq i \leq n$, such that no matter how we pick one set from each $\mathcal{F}_i$, they are always in convex position. Note that the exponent for c_n here is better than that of [5].

Recent research by Pach and Tóth [70] investigates Erdős-Szekeres type problems in which the points are replaced by convex sets that are not necessarily disjoint.

5.3. **Restricted Planar Point Sets.** The size of the coordinates of the points in the configurations given by Kalbfleisch and Stanton [51] that meet the conjectured upper bound on $N(n)$ grows very quickly. A step toward showing that this is unavoidable was taken by Alon *et al.* [1].

Suppose that X is a set of points in the plane, with no three on a line. Let $q(A)$ be the ratio of the largest distance between two points of X to the smallest distance between two points of X. The authors of [1] prove that if $|X| = k$ and $q(X) \leq \alpha\sqrt{k}$, then there is a constant β, depending on α, so that X contains a subset of size at least $\beta k^{\frac{1}{4}}$ in convex position. In other words, the restriction of the Erdős-Szekeres problem to point sets with relatively uniform distances between points yields a function $N(n)$ that is at most a fourth-degree polynomial.

The results of [1] were improved by Valtr [82]. Under the same conditions, $|X| = k$ and $q(X) \leq \alpha\sqrt{k}$, Valtr shows that there is a constant $\beta = \beta(\alpha)$ so that X contains a subset of size at least $\beta k^{\frac{1}{3}}$ in convex position. Furthermore, he constructs, for any $c > 5.96$ and k large enough, a set A_k of k points with no subset in convex position larger than $ck^{\frac{1}{3}}$. The sets A_k have the property that they contain no empty convex 7-gons.

5.4. **Polygons That Are Empty Modulo q.** Recall that Horton [46] proved the existence of arbitrarily large sets of points in general position in the plane that contain no empty convex 7-gons. In view of this, the following conjecture of [6] is surprising.

Conjecture 5.4. *For any two positive integers q and n, $n \geq 3$, there is a smallest positive integer $C(n,q)$ so that any set X of $C(n,q)$ points in general position in the plane contains a subset n points in convex position for which the number of points of X in its interior is divisible by q.*

Bialostocki *et al.* prove in [6] that the conjecture is true in the cases $n \equiv 2(\text{mod } q)$ and $n \geq q+3$. Extremely large upper bounds on $C(n,q)$ in both cases are obtained by the Ramsey theoretic argument.

Caro [15] found a better upper bound that also holds for a more general function. Let X be a set of points in general position in the plane, and let G be an abelian group. If w is a function from X to G and K is a subset of X in convex position, then K is said to have *zero-sum interior* modulo G if

$$\sum_{x \in interior K} w(x) = 0 \text{ (in } G).$$

Theorem 5.5 ([15])**.** *For any two integers n and q, $n \geq q+2$, there is an integer $E(n,q)$ satisfying the following conditions:*

(1) *Let X be a set of points in general position in the plane, and let G be an abelian group of order q. Assume $w : X \to G$. Then $|X| \geq E(n,q)$ implies that X contains a set of n points in convex position that has zero-sum interior.*

(2) *For a given q, one has $E(n,q) \leq 2^{c(q)n}$, where $c(q)$ depends only on q but not on n or the structure of G.*

Caro [15] speculates that the bound for $E(n,q)$ can be considerably improved when $n \geq q+2$. A recent result of Károlyi *et al.* [53] is that Conjecture 5.4 is true for $n \geq \frac{5}{6}q + O(1)$.

5.5. **Duality.** The Erdős-Szekeres problem has a dual one in terms of arrangements of lines in the plane. Attention to this equivalent problem was first drawn by Goodman and Pollack [37]. An arrangement of lines is called *simple* if no two of the lines are parallel and no three of them meet in a point. The dual problem is then to determine the smallest integer $N(n)$ so that every simple arrangement of $N(n)$ lines together with a point q not on any line contains a sub-arrangement of n lines for which the cell containing q is a convex n-gon.

One generalization is to consider the smallest integer $p(n)$ so that every simple arrangement of $p(n)$ lines contains a sub-arrangement of n lines determining a convex n-gon. Harborth and Möller [44] show that this problem is only interesting if the arrangements are considered to be in the projective plane. To do this, one identifies opposite unbounded cells of the arrangements. It is trivially true that $p(n) \leq N(n)$. It is shown in [44] that $p(6) = 9$ and $p(n) \geq 1 + 2^{\frac{k+1}{2}}$.

A second generalization is to replace the lines by pseudolines. Goodman and Pollack [36] conjectured that the inequality $N(n) \leq 2^{n-2} + 1$ holds even if "lines" in the dual Erdős-Szekeres problem are replaced by "pseudolines". If we denote by $N_{ps}(n)$ the analogous function for pseudolines, one can see that the arguments of [32] and [80] remain valid and show that $N_{ps}(n) \leq \binom{2n-5}{n-2} + 2$. A non-stretchable arrangement of 16 lines for which no subarrangement of 6 lines forms a polygon containing a specified point is given by Morris [62]. Harborth and Möller also ask if the function $p(n)$ of their problem is altered by substituting pseudolines for lines.

5.6. **Generalized Convexity.** Many of the known results on the Erdős-Szekeres problem have been proved using only some very simple combinatorial properties of the plane. It is natural to ask what the most general framework is for studying this problem. One such framework is that of generalized convexity (see the books by Soltan [75] and Van de Vel [87] for an overview of this topic).

We start with a finite set X and a collection $\mathcal{F}$ of subsets of X. The pair $(X, \mathcal{F})$ is called a *convexity* on X if the following hold:

1) $\emptyset$ and X are in $\mathcal{F}$,

2) $\mathcal{F}$ is closed under intersection.

For any subset A of X, define $co(A)$ to be the smallest member of $\mathcal{F}$ containing A. A subset A of X is called *convexly independent* if $a \notin co(A \setminus a)$ for all $a \in A$.

A set X of points in E^d is said to *realize* a convexity $(X, \mathcal{F})$ if $A \in \mathcal{F}$ precisely when $A = K \cap X$ for some convex subset K of E^d. If $(X, \mathcal{F})$ is realizable and satisfies a nondegeneracy assumption, then we have seen that there is a function $N(n)$ so that X contains a convexly independent set of size n whenever $|X| \geq N(n)$. For nondegeneracy, one simply stipulates that, for some k, all subsets of X of size at most k are convexly independent. One would like to replace the condition of realizability by a simpler combinatorial condition.

A convexity $(X, \mathcal{F})$ is said to have the *anti-exchange property* if for any subset A of X and $x, y \notin co(A)$, $x \in co(A \cup y)$ implies $y \notin co(A \cup x)$. Several names have been given to convexities with the anti-exchange property, most notably *convex geometry* (see [24]) and *antimatroid* (see [59]). Note that Coppel [19] uses the term convex geometry to refer to a different set of axioms.

A *basis* of a set $A \subseteq X$ is a minimal set $B \subseteq A$ such that $co(B) = co(A)$. The anti-exchange property is equivalent to the property that every set $A \subseteq X$ has a unique basis. The anti-exchange property by itself is not a strong enough property to provide a structure in which one can carry through Szekeres' Ramsey-theoretic proof of the Erdős-Szekeres theorem. We will see that the addition of one more property is sufficient for this purpose.

The *Carathéodory number* of a convexity $\mathcal{F}$ is the least positive integer c such that $co(Y) = \cup\{co(Z) : Z \subseteq Y, |Z| \leq c\}$ for any $Y \subseteq X$. A set $Y \subseteq X$ is said to be in *nice position* if any c points of Y are convexly independent.

Let c be the Carathéodory number of a convexity $(X, \mathcal{F})$. We say that $(X, \mathcal{F})$ satisfies the *simplex partition property* if for any set $\{z_1, z_2, \ldots, z_{c+2}\}$ of $c+2$ points in nice position, with $\{z_{c+1}, z_{c+2}\} \subseteq co(z_1, \ldots, z_c)$, the point z_{c+2} belongs to exactly one of the sets $co(z_1, \ldots, z_{i-1}, z_{c+1}, z_{i+1}, \ldots, z_c)$, $i = 1, \ldots, c$.

Lemma 5.6. *Let convexity $(X, \mathcal{F})$ satisfy the anti-exchange property and the simplex partition property. Any set of $c+2$ points of X in nice position contains $c+1$ convexly independent points.*

Proof. Let $\{z_1, \dots, z_{c+2}\}$ be a set of points of X in nice position. We may assume that $z_{c+2} \in co(z_2, z_3, \dots, z_{c+1})$ and $z_{c+1} \in co(z_1, z_3, z_4, \dots, z_c, z_{c+2})$. We then claim that $A = \{z_1, z_2, z_4, \dots, z_{c+2}\}$ is convexly independent. The simplex partition property implies that $z_i \notin co(A \backslash z_i)$ for $i = c+1, c+2$. The anti-exchange property implies that $z_i \notin co(A \backslash z_i)$ for $i = 1, 2, 4, \dots, c$. □

Theorem 5.7. *Let $(X, \mathcal{F})$ be a convexity with the anti-exchange property and the simplex partition property, and with Caratheodory number $c \geq 3$. Then for any positive integer n there exists a positive integer $N(n)$ such that any set $Y \subseteq X$ of $N(n)$ points in nice position contains n convexly independent points.*

Proof. Put $N(n) = R_{c+1}(n, c+2)$. □

Another variant of the Erdős-Szekeres theorem for convexities satisfying almost the same properties as the above theorem is given by Korte and Lovász [58].

If $(X, \mathcal{F})$ is a convexity, then a set $A \subseteq X$ is called *free* if it is both convexly independent and a member of $\mathcal{F}$. For realizable $(X, \mathcal{F})$, a free set is the set of vertices of an empty convex polytope. The *Helly number* of $(X, \mathcal{F})$ is the smallest integer h such that for any subfamily $\mathcal{B}$ of $\mathcal{F}$, if each h or fewer members of $\mathcal{B}$ have nonempty intersection, then the intersection of all members of $\mathcal{B}$ is nonempty. It is proved in [48] that the Helly number of a convexity is always at least as large as the cardinality of its largest free set. If the convexity satisfies the anti-exchange property, however, these two numbers are equal.

An interesting problem seems to be to find an infinite sequence of convexities $\{(X_i, \mathcal{F}_i)\}$ so that $|X_i| = i$, each $(X_i, \mathcal{F}_i)$ satisfies combinatorial conditions that "almost" imply realizability in general position in the plane, and no $(X_i, \mathcal{F}_i)$ contains a free set of cardinality 6.

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