HW08 Solutions

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Every non-x vert is in a C_4. All non-x verts have deg 2, so the y_1, y_2, y_3, y are in a C_4 and are not connected to anything else.

Statement of Prob 3

We use the language of 3-hypergraphs. One predicate: $E(x, y, z)$.
We assume E is symmetric.

$$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

If $(\exists N \ge X(n, m))[N \in \mathrm{spec}(\phi)]$ then

$$\{n + m, n + m + 1, \ldots\} \subseteq \mathrm{spec}(\phi).$$

Fill in the X and prove it.

SOL to Prob 3: Sets U, Y

Assume $\exists$ 3-hypergraph $G = (V, E)$ on $\geq X$ vertices, $G \models \phi$.
Witnesses: $u_1, \ldots, u_n$ be the witnesses.

$$U = \{u_1, \ldots, u_n\} \qquad Y = V - U \qquad |Y| = X - n = A.$$

$$Y = \{y_1, \ldots, y_A\}$$

Want Y superhomog.

SOL to Prob 3 Y and $\binom{U}{2}$

Map $y_i \in Y$ to the $\binom{n}{2}$ sized vector indexed by $\binom{[n]}{2}$:
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$\{y_1, \ldots, y_B\}$ have same rel to all pairs in $\binom{U}{2}$.

SOL to Prob 3: RECAP

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X TBD

$A = X - n$

$B = \frac{A}{2\binom{n}{2}}$

$\binom{Y}{2}$ and U

Have: $\{y_1, \ldots, y_B\}$ have same rel to all pairs in $\binom{U}{2}$.

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Form the following coloring $COL : \binom{Y}{2} \rightarrow [\{0,1\}^n]$.

$$COL(y_i, y_j) = (E(y_i, y_j, u_1), \ldots, E(y_i, y_j, u_n))$$

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Replace Y with the homog set. Re-index to get

$$Y = \{y_1, \ldots, y_C\}$$

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We will see how big C needs to be, then how big B needs to be.

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So what is X? Next Slide.

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$X = A + n = 2^{\binom{n}{2}} R_2(C, 2^n) = R_2(R_3(m, 2), 2^n) + n$.

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Let $L(c)$ be the least n (if it exists) so that for all c-colorings of $\{1, \ldots, n\}$ there exists two numbers that are the same color that are a Liam apart.

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1. Find an upper bound on $L(2)$.
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Contradiction.

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SOL to b (Diagram)

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More generally, $\mathrm{COL}(x) = \mathrm{COL}(x + 54)$.

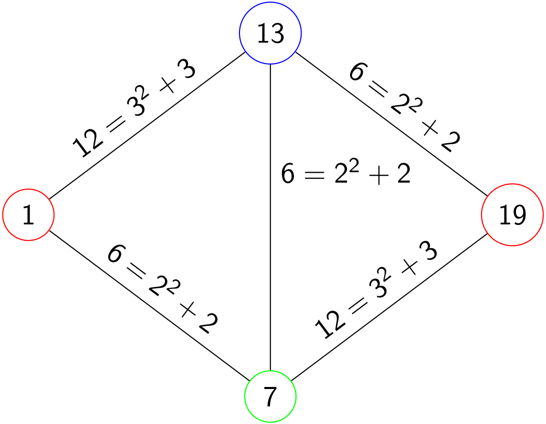

Figure: $\mathrm{COL}(x) = \mathrm{COL}(x + 18)$

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Can we do better? I do not know.