The Infinite Can Ramsey Thm: Mileti's Proof

William Gasarch-U of MD

Recap

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- One used 4-ary Ramsey and 1-d Can Ramsey.
- One used 3-ary Ramsey, 1-d Can Ram, and Maximal Sets.

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2. My interest: educational.
3. My interest: better bounds when finitized.
4. This finization has never been written up. Will be an extra credit project.

Min-Homog, Max-Homog, Rainbow

Def: Let $COL : \binom{N}{2} \rightarrow \omega$. Let $V \subseteq N$. Assume $a < b$ and $c < d$.

- V is *homog* if $COL(a,b) = COL(c,d)$ iff $TRUE$.
- V is *min-homog* if $COL(a,b) = COL(c,d)$ iff $a = c$.
- V is *max-homog* if $COL(a,b) = COL(c,d)$ iff $b = d$.
- V is *rainb* if $COL(a,b) = COL(c,d)$ iff $a = c$ and $b = d$.

Can Ramsey Thm for $\binom{N}{2}$: For all $COL : \binom{N}{2} \rightarrow \omega$, there exists an infinite set V such that either V is homog, min-homog, max-homog, or rainb.

Notation

$(\exists^\infty x \in A)$ means **for an infinite number of $x \in A$**

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$(\exists^\infty x \in A)$ means **for an infinite number of $x \in A$**

$(\forall^\infty x \in A)$ means **for all but a finite number of $x \in A$**

First Step of Construction

The following notation will make later cases similar to this case.

$V_1 = \mathsf{N}$

$x_1 = 1$

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  $\mathrm{COL}'(x_1) = (\mathrm{RB}, 1)$. Similar to proof of 1-ary Can Ramsey.

In both cases let
V_2 be the new infinite set.
x_2 be the least element of V_2.

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Have V_2 and x_2.
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- $(\forall c \in \omega)(\forall^\infty y \in V_2)[\mathrm{COL}(x_2, x) \neq c]$.
  - For every color c the set of y with $\mathrm{COL}(x_2, y) = c$ is finite. Kill duplicates so that $\mathrm{COL}(x_2, ?)$ are all different. New set is W. Will not be final V_3.
  - $\mathrm{COL}'(x_2) = (\mathrm{RB}, 1)$ if x_1 and x_2 are **similar**.
    $\mathrm{COL}'(x_2) = (\mathrm{RB}, 2)$ if x_1 and x_2 are **different**.
    See next slide.

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Really $\omega = \mathsf{N}$ so they are all numbers.

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- $\mathrm{COL}(x_1, w_3)$, $\mathrm{COL}(x_1, w_4)$, $\cdots$ are all different.
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1. $(\exists^\infty w \in W)[\mathrm{COL}(x_1, w) = \mathrm{COL}(x_2, w)]$. Then let $V_3 = \{w \in W : COL(x_1, w) = \mathrm{COL}(x_2, w)\}$.

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V_{i+1} will be infinite.

ωth Step, Case 1

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Case 1 H occurs inf often as 1st coordinate and

$$(\exists c_0 \in \omega)(\exists^{\infty} x \in X)[\mathrm{COL}'(x) = (\mathrm{H}, c_0)].$$

$$H = \{x \in X \colon \mathrm{COL}'(x) = (\mathrm{H}, c_0)\}$$

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H is homog of color c_0.

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Eliminate Duplicates to get

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where $\mathrm{COL}'(h_i) = (\mathrm{H}, c_i)$ with c_i's different.
H is min-homog.

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If neither happens then H only occurs finite often as 1st coordinate.

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Eliminate those finite x such that $\mathrm{COL}'(x) = (\mathrm{H}, ?)$.

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For Cases 3,4 assume $(\forall x \in X)[\mathrm{COL}'(x) = (\mathrm{RB}, ?)]$.

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Case 3 $(\exists i_0 \in \mathsf{N})(\exists^\infty x \in X)[\mathrm{COL}'(x) = (\mathrm{RB}, i_0)]$.

$$H = \{x \in X \colon \mathrm{COL}'(x) = (\mathrm{RB}, i_0)\}$$

ωth Step, Case 3

Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
$\mathrm{COL}'(x) \in \{(\mathrm{RB}, j) \colon j \in \mathsf{N}\}$.
Case 3 $(\exists i_0 \in \mathsf{N})(\exists^\infty x \in X)[\mathrm{COL}'(x) = (\mathrm{RB}, i_0)]$.

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H is max-homog.

ωth Step, Case 4

Recap We have $X = \{x_1, x_2, x_3, \ldots\}$

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If Case 1,2,3 do not occur then have:

ωth Step, Case 4

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Case 4
$(\forall x)(\forall^\infty i)[\mathrm{COL}'(x) \neq (\mathrm{RB}, i)]$. Eliminate Duplicates to get

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$$H = \{h_1, h_2, h_3, \ldots\}$$

where $\mathrm{COL}'(h_j) = (\mathrm{RB}, c_j)$ with c_j's different.

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where $\mathrm{COL}'(h_j) = (\mathrm{RB}, c_j)$ with c_j's different.
So where are we now?
Let $a < b < c$.

ωth Step, Case 4

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$\mathrm{COL}'(x) \in \{(\mathrm{RB}, j) : j \in \mathsf{N}\}$.
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So where are we now?
Let $a < b < c$.

- All of the edges out of h_a to the right, are different from each other.

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So is H a rainbow set?

ωth Step, Case 4

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$\mathrm{COL}'(x) \in \{(\mathrm{RB}, j) : j \in \mathsf{N}\}$.
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So where are we now?
Let $a < b < c$.

- All of the edges out of h_a to the right, are different from each other.
- $\mathrm{COL}(h_a, h_c) \neq \mathrm{COL}(h_b, h_c)$.

So is H a rainbow set?
No. Counterexample on next slide.

Countexample Due to Liam Gerst

$$\mathrm{COL} : \binom{N}{2} \to \omega$$

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- $\mathrm{COL}(a,c) \neq \mathrm{COL}(b,c)$.

ωth Step, Case 4 (cont)

Recap

$$H = \{h_1, h_2, h_3, \ldots\}$$

Let $a < b < c$.

ωth Step, Case 4 (cont)

Recap

$$H = \{h_1, h_2, h_3, \ldots\}$$

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- All of the edges out of h_a to the right are different from each other.

ωth Step, Case 4 (cont)

Recap

$$H = \{h_1, h_2, h_3, \ldots\}$$

Let $a < b < c$.

- All of the edges out of h_a to the right are different from each other.
- $\mathrm{COL}(h_a, h_c) \neq \mathrm{COL}(h_b, h_c)$.

Claim For all $i \in$ N, c a color, $\deg_c(h_i) \le 2$.
Proof Assume, BWOC that $\deg_c(h_i) \ge 3$.
Case 1 There two vertices x, y to the right of h_i such that $\mathrm{COL}(h_i, x) = \mathrm{COL}(h_i, y) = c$. This contradicts that all the edges coming out of h_i are different.
Case 2 There two vertices x, y to the left of h_i such that $\mathrm{COL}(x, h_i) = \mathrm{COL}(y, h_i) = c$. This contradicts that x and y disagree.
End of Proof of Claim

Last Step

Recall
Lemma Let X be infinite. Let $COL : \binom{X}{2} \rightarrow \omega$. Let $d \in \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \le d$ then there is an infinite rainb set.
We apply this to our set H with $d = 2$ to get a rainbow set.