The Infinite Can Ramsey Thm: Mileti’s Proof

William Gasarch-U of MD
Recap

We gave two proofs of Inf Can Ramsey:

▶ One used 4-ary Ramsey and 1-d Can Ramsey.
▶ One used 3-ary Ramsey, 1-d Can Ram, and Maximal Sets.

Is there a proof that is similar in spirit to the proof of Inf Ramsey?

Yes. It is due to Joseph Mileti.

1. His interest: He got a more constructive proof of Can Ramsey.
3. My interest: better bounds when finitized.
4. This finization has never been written up. Will be an extra credit project.
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Min-Homog, Max-Homog, Rainbow

**Def:** Let $COL : \binom{\mathbb{N}}{2} \to \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- $V$ is *homog* if $COL(a, b) = COL(c, d)$ iff TRUE.
- $V$ is *min-homog* if $COL(a, b) = COL(c, d)$ iff $a = c$.
- $V$ is *max-homog* if $COL(a, b) = COL(c, d)$ iff $b = d$.
- $V$ is *rainb* if $COL(a, b) = COL(c, d)$ iff $a = c$ and $b = d$.

**Can Ramsey Thm for $\binom{\mathbb{N}}{2}$:** For all $COL : \binom{\mathbb{N}}{2} \to \omega$, there exists an infinite set $V$ such that either $V$ is homog, min-homog, max-homog, or rainb.
(∃∞x ∈ A) means for an infinite number of x ∈ A
Notation

$(\exists^\infty x \in A)$ means for an infinite number of $x \in A$

$(\forall^\infty x \in A)$ means for all but a finite number of $x \in A$
First Step of Construction

The following notation will make later cases similar to this case.

\( V_1 = \mathbb{N} \)

\( x_1 = 1 \)

Have \( COL : \binom{V_1}{2} \rightarrow \omega \).
First Step of Construction

The following notation will make later cases similar to this case.

\( V_1 = \mathbb{N} \)

\( x_1 = 1 \)

Have \( \text{COL} : \binom{V_1}{2} \rightarrow \omega \).

One of the following happens:

\[ \exists c \in \omega \left( \exists \infty y \in V_1 \left[ \text{COL}(x_1, y) = c \right] \right) \]

Kill all those who disagree.

\( \text{COL}'(x_1) = (H, c) \).

Similar to 1st step of Inf Ramsey.

\[ \forall c \in \omega \left( \forall \infty y \in V_1 \left[ \text{COL}(x_1, y) \neq c \right] \right) \]

For every color \( c \), the set of \( y \) with \( \text{COL}(x_1, y) = c \) is finite.

Kill duplicates, so in new set \( \text{COL}'(x_1) \) are all different.

\( \text{COL}'(x_1) = (RB, 1) \).

Similar to proof of 1-ary Can Ramsey.

In both cases let \( V_2 \) be the new infinite set.

\( x_2 \) be the least element of \( V_2 \).
First Step of Construction

The following notation will make later cases similar to this case.

\( V_1 = N \)
\( x_1 = 1 \)

Have \( COL : (V_1) \rightarrow \omega \).

One of the following happens:

- \((\exists c \in \omega)(\exists \infty y \in V_1)[COL(x_1, y) = c]\).
First Step of Construction

The following notation will make later cases similar to this case.
\( V_1 = \mathbb{N} \)
\( x_1 = 1 \)
Have  \( COL : \binom{V_1}{2} \rightarrow \omega. \)
One of the following happens:

- (\( \exists c \in \omega \)(\( \exists \infty y \in V_1 \))\( COL(x_1, y) = c \).
  Kill all those who disagree.  \( COL'(x_1) = (H, c) \).
  Similar to 1st step of Inf Ramsey.
First Step of Construction

The following notation will make later cases similar to this case.

\[ V_1 = \mathbb{N} \]
\[ x_1 = 1 \]

Have \( COL : \binom{V_1}{2} \rightarrow \omega \).

One of the following happens:

- \((\exists c \in \omega)(\exists \infty y \in V_1)[COL(x_1, y) = c]\). Kill all those who disagree. \( COL'(x_1) = (H, c) \). Similar to 1st step of Inf Ramsey.

- \((\forall c \in \omega)(\forall \infty y \in V_1)[COL(x_1, y) \neq c]\). For every color \( c \) the set of \( y \) with \( COL(x_1, y) = c \) is finite.
First Step of Construction

The following notation will make later cases similar to this case.

\( V_1 = \mathbb{N} \)

\( x_1 = 1 \)

Have \( COL : \binom{V_1}{2} \to \omega \).

One of the following happens:

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- \((\forall c \in \omega)(\forall \infty y \in V_1)[COL(x_1, y) \neq c]\).
  For every color \( c \) the set of \( y \) with \( COL(x_1, y) = c \) is finite.
  Kill duplicates, so in new set \( COL'(x_1) = (RB, 1) \).
  Similar to proof of 1-ary Can Ramsey.
First Step of Construction

The following notation will make later cases similar to this case.

$V_1 = N$

$x_1 = 1$

Have $COL : (V_1^2) \rightarrow \omega$.

One of the following happens:

- $(\exists c \in \omega)(\exists \infty y \in V_1)[COL(x_1, y) = c]$. Kill all those who disagree. $COL'(x_1) = (H, c)$. Similar to 1st step of Inf Ramsey.

- $(\forall c \in \omega)(\forall \infty y \in V_1)[COL(x_1, y) \neq c]$. For every color $c$ the set of $y$ with $COL(x_1, y) = c$ is finite. Kill duplicates, so in new set $COL(x_1, ?)$ are all different. $COL'(x_1) = (RB, 1)$. Similar to proof of 1-ary Can Ramsey.

In both cases let

$V_2$ be the new infinite set.

$x_2$ be the least element of $V_2$. 
Second Step of Construction

Have $V_2$ and $x_2$.
Have $COL : \binom{V_2}{2} \rightarrow \omega$. 
Second Step of Construction

Have $V_2$ and $x_2$.

Have $COL : \left(\frac{V_2}{2}\right) \rightarrow \omega$.

$\exists c \in \omega \exists y \in V_2 [COL(x_2, y) = c]$. Then restrict to that set and color $x_2$ with $(H, c)$. Similar to 2nd step of Inf Ram.
Second Step of Construction

Have $V_2$ and $x_2$.
Have $\text{COL} : (\frac{V_2}{2}) \to \omega$.

▶ $(\exists c \in \omega)(\exists^\infty y \in V_2)[\text{COL}(x_2, y) = c]$. Then restrict to that set and color $x_2$ with $(H, c)$. Similar to 2nd step of Inf Ram.

▶ $(\forall c \in \omega)(\forall^\infty y \in V_2)[\text{COL}(x_2, x) \neq c]$. 

For every color $c$ the set of $y$ with $\text{COL}(x_2, y) = c$ is finite.
Kill duplicates so that $\text{COL}(x_2, ?)$ are all different. 
New set is $W$. Will not be final $V_3$.

▶ $\text{COL}'(x_2) = (RB, 1)$ if $x_1$ and $x_2$ are similar.

$\text{COL}'(x_2) = (RB, 2)$ if $x_1$ and $x_2$ are different.

See next slide.
Have $V_2$ and $x_2$.

Have $COL : \binom{V_2}{2} \to \omega$.

- $(\exists c \in \omega)(\exists \infty y \in V_2)[COL(x_2, y) = c]$. Then restrict to that set and color $x_2$ with $(H, c)$. Similar to 2nd step of Inf Ram.
- $(\forall c \in \omega)(\forall \infty y \in V_2)[COL(x_2, x) \neq c]$.
  - For every color $c$ the set of $y$ with $COL(x_2, y) = c$ is finite. Kill duplicates so that $COL(x_2, ?)$ are all different. New set is $W$. Will not be final $V_3$. 
Second Step of Construction

Have $V_2$ and $x_2$.
Have $COL : \binom{V_2}{2} \to \omega$.

- $(\exists c \in \omega)(\exists \infty y \in V_2)[COL(x_2, y) = c]$. Then restrict to that set and color $x_2$ with $(H, c)$. Similar to 2nd step of Inf Ram.
- $(\forall c \in \omega)(\forall \infty y \in V_2)[COL(x_2, x) \neq c]$.

- For every color $c$ the set of $y$ with $COL(x_2, y) = c$ is finite.
  Kill duplicates so that $COL(x_2, ?)$ are all different.
  New set is $W$. Will not be final $V_3$.
- $COL'(x_2) = (RB, 1)$ if $x_1$ and $x_2$ are similar.
  $COL'(x_2) = (RB, 2)$ if $x_1$ and $x_2$ are different.
See next slide.
Convention

When we say \((H, j)\) we think of \(j\) as a color.
We also say \(j \in \omega\).
Convention

When we say $(H, j)$ we think of $j$ as a color. We also say $j \in \omega$.

When we say $(RB, j)$ we think of $j$ as an index. We also say $j \in \mathbb{N}$. 

$\text{Really } \omega = \mathbb{N}$ so they are all numbers.
Convention

When we say \((H,j)\) we think of \(j\) as a color.
We also say \(j \in \omega\).

When we say \((RB,j)\) we think of \(j\) as an index.
We also say \(j \in \mathbb{N}\).

Really \(\omega = \mathbb{N}\) so they are all numbers.
\[ \text{COL}'(x_1), \text{COL}'(x_2) \in \{(RB, 1), (RB, 2)\} \]

\[ W = \{ w_3, w_4, \ldots, \} \]
COL′(x₁), COL′(x₂) ∈ {(RB, 1), (RB, 2)}

\[ W = \{ w_3, w_4, \ldots, \} \]

Note following
- COL(x₁, w₃), COL(x₁, w₄), ⋯ are all different.
- COL(x₂, w₃), COL(x₂, w₄), ⋯ are all different.
COL'(x_1), COL'(x_2) \in \{(RB, 1), (RB, 2)\}

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Note following
- COL(x_1, w_3), COL(x_1, w_4), \ldots \text{ are all different.}
- COL(x_2, w_3), COL(x_2, w_4), \ldots \text{ are all different.}

One of the following occurs.
COL’(x₁), COL’(x₂) ∈ {(RB, 1), (RB, 2)}

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- COL(x₁, w₃), COL(x₁, w₄), ⋯ are all different.
- COL(x₂, w₃), COL(x₂, w₄), ⋯ are all different.

One of the following occurs.

1. \( \exists \infty w \in W \)[COL(x₁, w) = COL(x₂, w)]. Then let \( V₃ = \{ w \in W : \text{COL}(x₁, w) = \text{COL}(x₂, w) \} \).
\( \text{COL}'(x_1), \text{COL}'(x_2) \in \{(\text{RB}, 1), (\text{RB}, 2)\} \)

\[ W = \{w_3, w_4, \ldots, \} \]

Note following

- \( \text{COL}(x_1, w_3), \text{COL}(x_1, w_4), \cdots \) are all different.
- \( \text{COL}(x_2, w_3), \text{COL}(x_2, w_4), \cdots \) are all different.

One of the following occurs.

1. \((\exists \infty w \in W)[\text{COL}(x_1, w) = \text{COL}(x_2, w)]\). Then let \( V_3 = \{w \in W : \text{COL}(x_1, w) = \text{COL}(x_2, w)\} \).
   \( \text{COL}'(x_2) = (\text{RB}, 1) \).
   Note that \((\forall y \in V_3)[\text{COL}(x_1, y) = \text{COL}(x_2, y)] \& |V_3| = \infty\)
\(\text{COL}'(x_1), \text{COL}'(x_2) \in \{(\text{RB}, 1), (\text{RB}, 2)\}\)

\[W = \{w_3, w_4, \ldots, \}\]

Note following

\[\text{COL}(x_1, w_3), \text{COL}(x_1, w_4), \ldots\text{ are all different.}\]

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1. \((\exists^\infty w \in W)[\text{COL}(x_1, w) = \text{COL}(x_2, w)]\). Then let

\[V_3 = \{w \in W: \text{COL}(x_1, w) = \text{COL}(x_2, w)\}\]

\[\text{COL}'(x_2) = (\text{RB}, 1)\]

Note that \((\forall y \in V_3)[\text{COL}(x_1, y) = \text{COL}(x_2, y)]\ & |V_3| = \infty\]

2. \((\exists^\infty w \in W)[\text{COL}(x_1, w) \neq \text{COL}(x_2, w)]\). Then let

\[V_3 = \{w \in W: \text{COL}(x_1, w) \neq \text{COL}(x_2, w)\}\].
COL′(x₁), COL′(x₂) ∈ {(RB, 1), (RB, 2)}

\[ W = \{ w₃, w₄, \ldots, \} \]

Note following

- COL(x₁, w₃), COL(x₁, w₄), \cdots are all different.
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One of the following occurs.

1. \((∃∞ w ∈ W)[COL(x₁, w) = COL(x₂, w)]\). Then let \(V₃ = \{ w ∈ W : COL(x₁, w) = COL(x₂, w) \}\).
   COL′(x₂) = (RB, 1).
   Note that \((∀y ∈ V₃)[COL(x₁, y) = COL(x₂, y)] \& |V₃| = ∞\)

2. \((∃∞ w ∈ W)[COL(x₁, w) \neq COL(x₂, w)]\). Then let \(V₃ = \{ w ∈ W : COL(x₁, w) \neq COL(x₂, w) \}\).
   COL′(x₂) = (RB, 2).
   Note that \((∀y ∈ V₃)[COL(x₁, y) \neq COL(x₂, y)] \& |V₃| = ∞\)
Third Step, \( i \)th Step

\( V_3 \) is defined and is infinite. \( x_1, x_2 \) are colored. \( x_3 \) is least element of \( V_3 \).
Third Step, \textit{ith} Step

$V_3$ is defined and is infinite. $x_1, x_2$ are colored. \\
$x_3$ is least element of $V_3$. \\
HW: Do third step.
Third Step, $i$th Step

$V_3$ is defined and is infinite. $x_1, x_2$ are colored.
$x_3$ is least element of $V_3$.
HW: Do third step.
After third step
$\text{COL}'(x_3) \in \{(H, j): j \in \omega\} \cup \{(\text{RB}, j): j \leq 3\}$.
$V_4$ will be infinite.
Third Step, \(i\)th Step

\(V_3\) is defined and is infinite. \(x_1, x_2\) are colored.
\(x_3\) is least element of \(V_3\).
HW: Do third step.
After third step
\(\text{COL}'(x_3) \in \{(H,j) : j \in \omega\} \cup \{(\text{RB},j) : j \leq 3\}\).
\(V_4\) will be infinite.

\(V_i\) is defined and is infinite. \(x_1, \ldots, x_{i-1}\) are colored.
\(x_i\) is least element of \(V_i\).
Third Step, $i$th Step

$V_3$ is defined and is infinite. $x_1, x_2$ are colored.

$x_3$ is least element of $V_3$.

HW: Do third step.

After third step

$COL'(x_3) \in \{(H, j): j \in \omega\} \cup \{(RB, j): j \leq 3\}$.

$V_4$ will be infinite.

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HW: Do $i$th step.
Third Step, $i$th Step

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HW: Do third step.
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$\text{COL}'(x_3) \in \{(H, j): j \in \omega\} \cup \{(\text{RB}, j): j \leq 3\}$.
$V_4$ will be infinite.

$V_i$ is defined and is infinite. $x_1, \ldots, x_{i-1}$ are colored. $x_i$ is least element of $V_i$.
HW: Do $i$th step.
After $i$th step
$\text{COL}'(x_i) \in \{(H, j): j \in \omega\} \cup \{(\text{RB}, j): j \leq i\}$.
$V_{i+1}$ will be infinite.
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
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For all $x \in X$
$\text{COL}'(x) \in \{(H, j): j \in \omega\} \cup \{(RB, j): j \in N\}$. 
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
For all $x \in X$
$COL'(x) \in \{(H, j): j \in \omega\} \cup \{(RB, j): j \in \mathbb{N}\}$.
Key We started with $COL: \binom{\mathbb{N}}{2} \rightarrow \omega$ and now have $COL': X \rightarrow \omega$. 
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
For all $x \in X$
$\text{COL}'(x) \in \{(H, j): j \in \omega\} \cup \{(\text{RB}, j): j \in \mathbb{N}\}$.
Key We started with $\text{COL}: (\mathbb{N})_2 \rightarrow \omega$ and now have $\text{COL}' : X \rightarrow \omega$.
Case 1 $H$ occurs inf often as 1st coordinate and
\[(\exists c_0 \in \omega)(\exists^\infty x \in X)[\text{COL}'(x) = (H, c_0)].\]
\[H = \{x \in X: \text{COL}'(x) = (H, c_0)\}\]
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
For all $x \in X$
$\text{COL}'(x) \in \{(H, j) : j \in \omega\} \cup \{(\text{RB}, j) : j \in \mathbb{N}\}$.

Key We started with $\text{COL}: \binom{\mathbb{N}}{2} \to \omega$ and now have $\text{COL}' : X \to \omega$.

Case 1 $H$ occurs inf often as 1st coordinate and

$(\exists c_0 \in \omega)(\exists^\infty x \in X)[\text{COL}'(x) = (H, c_0)].$

$H = \{x \in X : \text{COL}'(x) = (H, c_0)\}$

COL restricted to $\binom{H}{2}$ is always color $c_0$. 
Recap We have \( X = \{x_1, x_2, x_3, \ldots\} \)
For all \( x \in X \)
\( \text{COL}'(x) \in \{(H, j): j \in \omega\} \cup \{(\text{RB}, j): j \in \mathbb{N}\} \).

Key We started with \( \text{COL}: \mathbb{N}^2 \to \omega \) and now have
\( \text{COL}' : X \to \omega \).

Case 1 \( H \) occurs inf often as 1st coordinate and

\[(\exists c_0 \in \omega)(\exists \infty x \in X)[\text{COL}'(x) = (H, c_0)].\]

\[ H = \{x \in X : \text{COL}'(x) = (H, c_0)\} \]

\( \text{COL} \) restricted to \( \binom{H}{2} \) is always color \( c_0 \).
\( H \) is homog of color \( c_0 \).
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
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$\text{COL}'(x) \in \{(H, j) : j \in \omega\} \cup \{(RB, j) : j \in \mathbb{N}\}$. 
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
\[\text{COL}'(x) \in \{(H, j): j \in \omega\} \cup \{(\text{RB}, j): j \in \mathbb{N}\}.\]

Case 2 $H$ occurs inf often as 1st coordinate and
\[(\forall c)(\forall^\infty x)[\text{COL}'(x) \neq (H, c)].\]

Eliminate Duplicates to get
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
$\text{COL}'(x) \in \{(H, j) : j \in \omega\} \cup \{(\text{RB}, j) : j \in \mathbb{N}\}$.

Case 2 $H$ occurs inf often as 1st coordinate and

$$(\forall c)(\forall^\infty x)[\text{COL}'(x) \neq (H, c)].$$

Eliminate Duplicates to get

$$H = \{h_1, h_2, h_3, \ldots\}$$

where $\text{COL}'(h_i) = (H, c_i)$ with $c_i$'s different.
Recap We have \(X = \{x_1, x_2, x_3, \ldots\}\)
\(\text{COL}'(x) \in \{(H, j): j \in \omega\} \cup \{(RB, j): j \in \mathbb{N}\}\).

Case 2 \(H\) occurs infinitely often as 1st coordinate and
\[
(\forall c)(\forall^\infty x)[\text{COL}'(x) \neq (H, c)].
\]

Eliminate Duplicates to get
\[
H = \{h_1, h_2, h_3, \ldots\}
\]
where \(\text{COL}'(h_i) = (H, c_i)\) with \(c_i\)'s different.
\(H\) is min-homog.
If Cases 1, 2 Do Not Occur Then . . .
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Case 1 $H$ occurs inf often as 1st coordinate and

$$(\exists c_0 \in \omega)(\exists^{\infty} x \in X)[\text{COL}'(x) = (H, c_0)].$$
If Cases 1,2 Do Not Occur Then . . .

Case 1 \( H \) occurs \( \infty \) often as 1st coordinate and

\[
(\exists c_0 \in \omega)(\exists^\infty x \in X) [\text{COL}'(x) = (H, c_0)].
\]

Case 2 \( H \) occurs \( \infty \) often as 1st coordinate and

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(\forall c)(\forall^\infty x) [\text{COL}'(x) \neq (H, c)].
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If Cases 1,2 Do Not Occur Then . . .

Case 1  $H$ occurs inf often as 1st coordinate and

$$(\exists c_0 \in \omega)(\exists^\infty x \in X)[\text{COL}'(x) = (H, c_0)].$$

Case 2  $H$ occurs inf often as 1st coordinate and

$$(\forall c)(\forall^\infty x)[\text{COL}'(x) \neq (H, c)].$$

If neither happens then $H$ only occurs finite often as 1st coordinate.
If Cases 1, 2 Do Not Occur Then . . .

Case 1 $H$ occurs inf often as 1st coordinate and

$$(\exists c_0 \in \omega)(\exists^\infty x \in X)[\text{COL}'(x) = (H, c_0)].$$

Case 2 $H$ occurs inf often as 1st coordinate and

$$(\forall c)(\forall^\infty x)[\text{COL}'(x) \neq (H, c)].$$

If neither happens then $H$ only occurs finite often as 1st coordinate. Eliminate those finite $x$ such that $\text{COL}'(x) = (H, ?)$. 
If Cases 1, 2 Do Not Occur Then . . .

**Case 1** $H$ occurs inf often as 1st coordinate and

$$(\exists c_0 \in \omega)(\exists^\infty x \in X)[\text{COL}'(x) = (H, c_0)].$$

**Case 2** $H$ occurs inf often as 1st coordinate and

$$(\forall c)(\forall^\infty x)[\text{COL}'(x) \neq (H, c)].$$

If neither happens then $H$ only occurs finite often as 1st coordinate. Eliminate those finite $x$ such that $\text{COL}'(x) = (H, ?)$. Keep the name of the set $X$ too avoid to much notation.
If Cases 1,2 Do Not Occur Then . . .

**Case 1** $H$ occurs inf often as 1st coordinate and

$$(\exists c_0 \in \omega)(\exists^\infty x \in X)[\text{COL}'(x) = (H, c_0)].$$

**Case 2** $H$ occurs inf often as 1st coordinate and

$$(\forall c)(\forall^\infty x)[\text{COL}'(x) \neq (H, c)].$$

If neither happens then $H$ only occurs finite often as 1st coordinate. Eliminate those finite $x$ such that $\text{COL}'(x) = (H, ?)$. Keep the name of the set $X$ too avoid to much notation. For Cases 3,4 assume $(\forall x \in X)[\text{COL}'(x) = (\text{RB}, ?)].$
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
Recap We have \( X = \{x_1, x_2, x_3, \ldots \} \)
\( COL'(x) \in \{(RB, j) : j \in \mathbb{N}\} \).
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
$\text{COL}'(x) \in \{(\text{RB}, j): j \in \mathbb{N}\}$.
Case 3 $(\exists i_0 \in \mathbb{N})(\exists x \in X)[\text{COL}'(x) = (\text{RB}, i_0)]$.

$$H = \{x \in X : \text{COL}'(x) = (\text{RB}, i_0)\}$$
$\omega$th Step, Case 3

Recap We have $X = \{x_1, x_2, x_3, \ldots\}$

$\text{COL}'(x) \in \{(RB, j) : j \in \mathbb{N}\}$.

Case 3 $(\exists i_0 \in \mathbb{N})(\exists x \in X)[\text{COL}'(x) = (RB, i_0)]$.

$$H = \{x \in X : \text{COL}'(x) = (RB, i_0)\}$$

$H$ is max-homog.
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$
$\text{COL'}(x) \in \{(RB, j) : j \in \mathbb{N}\}$.
Recap  We have $X = \{x_1, x_2, x_3, \ldots\}$
$\text{COL}'(x) \in \{(\text{RB}, j) : j \in \mathbb{N}\}$.
If Case 1,2,3 do not occur then have:
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$

$\text{COL}'(x) \in \{(\text{RB}, j): j \in \mathbb{N}\}$.

If Case 1, 2, 3 do not occur then have:

Case 4

$(\forall x)(\forall \infty i)[\text{COL}'(x) \neq (\text{RB}, i)]$. Eliminate Duplicates to get
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$  
$COL'(x) \in \{(RB, j) : j \in \mathbb{N}\}$.  
If Case 1,2,3 do not occur then have:  
Case 4  
$(\forall x)(\forall \infty i)[COL'(x) \neq (RB, i)]$. Eliminate Duplicates to get  
$$H = \{h_1, h_2, h_3, \ldots\}$$  
where $COL'(h_j) = (RB, c_j)$ with $c_j$’s different.
Recap We have \( X = \{x_1, x_2, x_3, \ldots \} \)
\( \text{COL}'(x) \in \{(RB, j) : j \in \mathbb{N}\} \).
If Case 1,2,3 do not occur then have:
Case 4

\( (\forall x)(\forall \infty i)[\text{COL}'(x) \neq (RB, i)] \). Eliminate duplicates to get

\[ H = \{h_1, h_2, h_3, \ldots \} \]

where \( \text{COL}'(h_j) = (RB, c_j) \) with \( c_j \)'s different.
So where are we now?
Let \( a < b < c \).
Recap We have \( X = \{x_1, x_2, x_3, \ldots\} \)
\( \text{COL}'(x) \in \{(\text{RB}, j) : j \in \mathbb{N}\} \).

If Case 1,2,3 do not occur then have:

Case 4

\((\forall x)(\forall \infty i)[\text{COL}'(x) \neq (\text{RB}, i)]\). Eliminate Duplicates to get

\[ H = \{h_1, h_2, h_3, \ldots\} \]

where \( \text{COL}'(h_j) = (\text{RB}, c_j) \) with \( c_j \)'s different.

So where are we now?

Let \( a < b < c \).

- All of the edges out of \( h_a \) to the right, are different from each other.
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$

$\text{COL}'(x) \in \{(RB, j) : j \in \mathbb{N}\}$.

If Case 1,2,3 do not occur then have:

Case 4

$(\forall x)(\forall \infty i)[\text{COL}'(x) \neq (RB, i)]$. Eliminate Duplicates to get

$$H = \{h_1, h_2, h_3, \ldots\}$$

where $\text{COL}'(h_j) = (RB, c_j)$ with $c_j$'s different.

So where are we now?

Let $a < b < c$.

- All of the edges out of $h_a$ to the right, are different from each other.
- $\text{COL}(h_a, h_c) \neq \text{COL}(h_b, h_c)$. 

No. Counterexample on next slide.
Recap We have $X = \{x_1, x_2, x_3, \ldots\}$

$\text{COL}'(x) \in \{(\text{RB}, j) : j \in \mathbb{N}\}$.

If Case 1, 2, 3 do not occur then have:

**Case 4**

$(\forall x)(\forall \infty i)[\text{COL}'(x) \neq (\text{RB}, i)]$. Eliminate Duplicates to get

$$H = \{h_1, h_2, h_3, \ldots\}$$

where $\text{COL}'(h_j) = (\text{RB}, c_j)$ with $c_j$’s different.

So where are we now?

Let $a < b < c$.

- All of the edges out of $h_a$ to the right, are different from each other.
- $\text{COL}(h_a, h_c) \neq \text{COL}(h_b, h_c)$.

So is $H$ a rainbow set?
Recap  We have \( X = \{x_1, x_2, x_3, \ldots \} \)
\( \text{COL}'(x) \in \{(\text{RB}, j): j \in \mathbb{N}\} \).
If Case 1,2,3 do not occur then have:

Case 4

\[(\forall x)(\forall i)[\text{COL}'(x) \neq (\text{RB}, i)]\]. Eliminate Duplicates to get

\[
H = \{h_1, h_2, h_3, \ldots \}
\]

where \( \text{COL}'(h_j) = (\text{RB}, c_j) \) with \( c_j \)'s different.

So where are we now?
Let \( a < b < c \).

- All of the edges out of \( h_a \) to the right, are different from each other.
- \( \text{COL}(h_a, h_c) \neq \text{COL}(h_b, h_c) \).

So is \( H \) a rainbow set?
No. Counterexample on next slide.
Countexample Due to Liam Gerst

\[ \text{COL} : \binom{N}{2} \rightarrow \omega \]
Countexample Due to Liam Gerst

\[ \text{COL} : \binom{N}{2} \to \omega \]

\[ \text{COL}(i, j) = |i - j| \]

Let \( a < b < c \).
Countexample Due to Liam Gerst

\[ \text{COL} : \binom{N}{2} \rightarrow \omega \]

\[ \text{COL}(i, j) = |i - j| \]

Let \( a < b < c \).

- All of the edges out of \( a \) to the right are different from each other.
Countexample Due to Liam Gerst

\[
\text{COL} : \binom{N}{2} \to \omega
\]

\[
\text{COL}(i, j) = |i - j|
\]

Let \( a < b < c \).

- All of the edges out of \( a \) to the right are different from each other.
- \( \text{COL}(a, c) \neq \text{COL}(b, c) \).
Recap

\[ H = \{ h_1, h_2, h_3, \ldots \} \]

Let \( a < b < c \).
ωth Step, Case 4 (cont)

Recap

\[ H = \{ h_1, h_2, h_3, \ldots \} \]

Let \( a < b < c \).

- All of the edges out of \( h_a \) to the right are different from each other.
Recap

\[ H = \{ h_1, h_2, h_3, \ldots \} \]

Let \( a < b < c \).

- All of the edges out of \( h_a \) to the right are different from each other.
- \( \text{COL}(h_a, h_c) \neq \text{COL}(h_b, h_c) \).

**Claim** For all \( i \in \mathbb{N} \), \( c \) a color, \( \deg_c(h_i) \leq 2 \).

**Proof** Assume, BWOC that \( \deg_c(h_i) \geq 3 \).

**Case 1** There two vertices \( x, y \) to the right of \( h_i \) such that \( \text{COL}(h_i, x) = \text{COL}(h_i, y) = c \). This contradicts that all the edges coming out of \( h_i \) are different.

**Case 2** There two vertices \( x, y \) to the left of \( h_i \) such that \( \text{COL}(x, h_i) = \text{COL}(y, h_i) = c \). This contradicts that \( x \) and \( y \) disagree.

**End of Proof of Claim**
Recall

**Lemma** Let $X$ be infinite. Let $COL : \binom{X}{2} \to \omega$. Let $d \in \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \leq d$ then there is an infinite rainbow set.

We apply this to our set $H$ with $d = 2$ to get a rainbow set.