The Muffin Problem

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How it Began

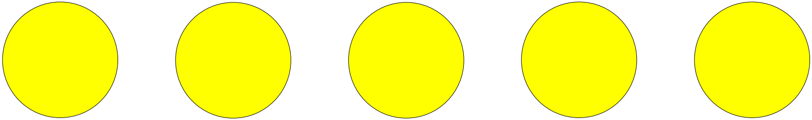
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

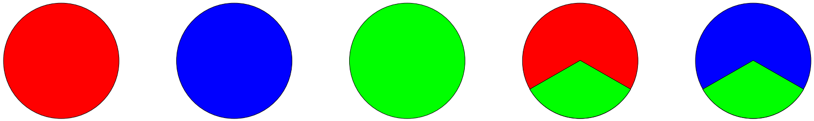
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

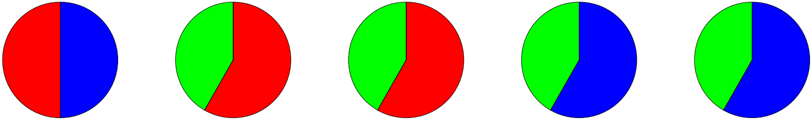
Work on it with your neighbor

Five Muffins, Three People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Carol	GREEN	$\frac{5}{12}+\frac{5}{12}+\frac{5}{12}+\frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

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5 Muffins, 3 People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

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(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

What Happened Next?

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Yada Yada Yada- in 2020:

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General Problem

$f(m,s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3) = \frac{5}{12}$ here.

We have two proofs that shown $f(m,s)$ exists, is rational, and is computable.
One use Linear Programming.
One use Integer Programming.

Amazing Results!/Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant !

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Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

What if $m < s$?

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Duality Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.

What if $m < s$?

Duality Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.
Hence we will just look at $m > s$.

Floor-Ceiling Thm Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le \mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

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Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

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Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{\mathbf{m}}{\mathbf{s\lceil 2m/s\rceil}}$.

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Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

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Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

FC Gives Upper Bound

Give m, s:

$$\mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\left\lceil 2m/s\right\rceil}, 1 - \frac{m}{s\left\lfloor 2m/s\right\rfloor}\right\}\right\}$$

Gives an upper bound. So we know

$$(\forall m,s)[f(m,s) \le \mathrm{FC}(m,s)].$$

Is the following true?

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No: If so my book would be about 20 pages.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1.$

$f(3k+1,3) = \frac{3k-1}{6k},\ k \geq 1.$

$f(3k+2,3) = \frac{3k+2}{6k+6}.$

Note: A Mod 3 Pattern.
Theorem: For all $m \geq 3$, $f(m,3) = \mathrm{FC}(m,3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$f(4k, 4) = 1$ (easy)

$f(1, 4) = \frac{1}{4}$ (easy)

$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = \mathrm{FC}(m, 4)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k+2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \mathrm{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k+3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k+4, 5) = \frac{5k+1}{10k+5}$
Note: A Mod 5 Pattern.
Theorem: For all $m \geq 5$ **except m=11**, $f(m, 5) = \mathrm{FC}(m, 5)$.

What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11,5) \geq \frac{13}{30}$.
2. $f(11,5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11,5) \leq \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

Options:

1. $f(11,5) = \frac{11}{25}$. Need to find procedure.
2. $f(11,5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11,5)$ in between. Need to find both.
4. $f(11,5)$ unknown to science!

Vote

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Vote WE SHOW $f(11,5) = \frac{13}{30}$. **Exciting** new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M.
The *other piece* from muffin M is the **buddy of x**.

Note that the **buddy** of x is of size

$$1 - x.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

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(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \le \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\ge \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

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That pieces **buddy** is of size:

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

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(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

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$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

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$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
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$f(11,5)=\frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets w, x, y, z and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

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Let x be the largest of x, y, z

$$x \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

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GREAT! This is where $\frac{13}{30}$ comes from!

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

HALF Method

The above reasoning can be used to *verify* that $f(11, 5) \leq \frac{13}{30}$ but could not *generate* the upper bound $\frac{13}{30}$.

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Can modify the method so that we have an easily computable function $\text{HALF}(m, s)$ such that

$$(\forall m, s)[f(m, s) \leq \min\{\text{FC}(m, s), \text{HALF}(m, s)\}]$$

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For $f(24,11)$ it fails!

$f(24, 11) \le \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\le \frac{19}{44}$.

$f(24,11) \le \frac{19}{44}$

Assume $(24,11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
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Case 1: A student gets ≥ 6 shares. Some piece $\le \frac{24}{11\times 6} < \frac{19}{44}$.

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Case 2: A student gets ≤ 3 shares. Some piece $\ge \frac{24}{11\times 3} = \frac{8}{11}$.
Buddy of that piece $\le 1 - \frac{8}{11} \le \frac{3}{11} < \frac{19}{44}$.

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Buddy of that piece $\le 1 - \frac{8}{11} \le \frac{3}{11} < \frac{19}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.
5-students: a student who gets 5 shares. s_5 is the number of them.

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$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

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$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.
$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: $\exists$ a share $\geq \frac{25}{44}$. Its **buddy** is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

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Case 3.2: There is a share $\leq \frac{19}{44}$. Duh.
Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44}\right)$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

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Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces A, B, C, D, E and $E \geq \frac{20}{44}$.

Since $A + B + C + D + E = \frac{24}{11}$ and $E > \frac{20}{44}$

$$A + B + C + D < \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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Assume A is the smallest of A, B, C, D.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

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Henceforth we assume all 5-shares are in

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Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume that Alice 4 pieces A, B, C, D and $D \le \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \le \frac{21}{44}$

$$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

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Proof: Assume that Alice 4 pieces A, B, C, D and $D \le \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \le \frac{21}{44}$

$$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume A is the largest of A, B, C.

$$A \ge \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The **buddy** of A is of size

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Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume that Alice 4 pieces A, B, C, D and $D \le \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \le \frac{21}{44}$

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Henceforth we assume all 4-shares are in

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Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

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$$\underset{\frac{19}{44}}{(} \quad ??\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0\ \text{shs} \quad \underset{\frac{21}{44}}{](} \quad ??\ \text{4-shs} \quad \underset{\frac{25}{44}}{)}$$

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Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.
Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

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Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.
Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\underset{\frac{19}{44}}{(} \quad 20\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0\ \text{shs} \quad \underset{\frac{21}{44}}{](} \quad 28\ \text{4-shs} \quad \underset{\frac{25}{44}}{)}$$

More Refined Picture of What is Going On

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Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

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If there was such a share then its **buddy** is in $[\frac{20}{44}, \frac{21}{44}]$.

More Refined Picture of What is Going On

($\frac{19}{44}$	20 5-shs	)[$\frac{20}{44}$	0 shs	]($\frac{21}{44}$	28 4-shs	) $\frac{25}{44}$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then its **buddy** is in $[\frac{20}{44}, \frac{21}{44}]$.
The following picture captures what we know so far.

($\frac{19}{44}$	20 5-shs	)[$\frac{20}{44}$	0	]($\frac{21}{44}$	8 S4-shs	)[$\frac{23}{44}$	0	]($\frac{24}{44}$	20 L4-shs	) $\frac{25}{44}$

S4= Small 4-shares
L4= Large 4-shares. L4 shares, 5-share: **buddies**, so |L4|=20.

$$\underset{\frac{19}{44}}{(}\quad 20\ \text{5-shs}\quad \underset{\frac{20}{44}}{)[}\quad 0\quad \underset{\frac{21}{44}}{](}\quad 8\ \text{S4-shs}\quad \underset{\frac{23}{44}}{)[}\quad 0\quad \underset{\frac{24}{44}}{](}\quad 20\ \text{L4-shs}\quad \underset{\frac{25}{44}}{)}$$

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{23}{44}$		$\frac{24}{44}$		$\frac{25}{44}$

Claim 2: Every 4-student has at least 3 L4 shares.

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Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

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Contradiction: Each 4-student gets ≥ 3 L4 shares. There are $s_4 = 7$ 4-students. Hence there are ≥ 21 L4-shares. But there are only 20.

INT Technique

The above reasoning can be used to *verify* that $f(24, 11) \leq \frac{19}{44}$ but could not *generate* the upper bound $\frac{19}{44}$.

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For $f(31, 19)$ it fails!

$f(31, 19) \le \frac{54}{133}$

We show $f(31, 19) \le \frac{54}{133}$.
Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$.

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By INT-technique methods obtain:
$s_3 = 14$, $s_4 = 5$.

(	20 4-shs	)[	0	](	S3 shs	)[	0	](	20 L3-shs	)
$\frac{54}{133}$		$\frac{55}{133}$		$\frac{59}{133}$		$\frac{74}{133}$		$\frac{78}{133}$		$\frac{79}{133}$

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We just look at the 3-shares:

$$\underset{\frac{59}{133}}{(} \quad \text{S3 shs} \quad \underset{\frac{74}{133}}{)[} \quad 0 \quad \underset{\frac{78}{133}}{](} \quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

$f(31,19) \le \frac{54}{133}$

(	S3 shs	)[	0	](	20 L3-shs	)
$\frac{59}{133}$		$\frac{74}{133}$		$\frac{78}{133}$		$\frac{79}{133}$

$f(31,19) \le \frac{54}{133}$

$$\underset{\frac{59}{133}}{(} \quad \text{S3 shs} \quad \underset{\frac{74}{133}}{)[} \quad 0 \quad \underset{\frac{78}{133}}{](} \quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ $(|J_1| = |J_2|)$
3. $J_3 = (\frac{78}{133}, \frac{79}{133})$ $(|J_3| = 20)$

Note: Split the shares of size 66.5 between J_1 and J_2.

Notation: An $e(1,1,3)$ students is a student who has a J_1-share, a J_1-share, and a J_3-share.

Generalize to $e(i,j,k)$ easily.

$f(31, 19) \leq \frac{54}{133}$

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1) Only students allowed: $e(1,2,3)$, $e(1,3,3)$, $e(2,2,2)$, $e(2,2,3)$. All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

$f(31, 19) \leq \frac{54}{133}$

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1) Only students allowed: $e(1, 2, 3)$, $e(1, 3, 3)$, $e(2, 2, 2)$, $e(2, 2, 3)$.
All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

2) No shares in $[\frac{61}{133}, \frac{64}{133}]$. Look at J_1-shares:
An $e(1, 2, 3)$-student has J_1-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
An $e(1, 3, 3)$-student has J_1-share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$.

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An $e(1,2,3)$-student has J_1-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
An $e(1,3,3)$-student has J_1-share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$.

3) No shares in $[\frac{69}{133}, \frac{72}{133}]$: $x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}]$.

$f(31,19) \leq \frac{54}{133}$

1. $J_1 = (\frac{59}{133}, \frac{61}{133})$
2. $J_2 = (\frac{64}{133}, \frac{66.5}{133})$
3. $J_3 = (\frac{66.5}{133}, \frac{69}{133})$ $(|J_2| = |J_3|)$
4. $J_4 = (\frac{72}{133}, \frac{74}{133})$ $(|J_1| = |J_4|)$
5. $J_5 = (\frac{78}{133}, \frac{79}{133})$ $(|J_5| = 20)$

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5. $J_5 = (\frac{78}{133}, \frac{79}{133})$ $(|J_5| = 20)$

The following are the only students who are allowed.
$e(1,5,5)$.
$e(2,4,5)$,
$e(3,4,5)$.
$e(4,4,4)$.

$f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be x
$e(2, 4, 5)$. Let the number of such students be y_1
$e(3, 4, 5)$. Let the number of such students be y_2.
$e(4, 4, 4)$. Let the number of such students be z.

$f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be x
$e(2, 4, 5)$. Let the number of such students be y_1
$e(3, 4, 5)$. Let the number of such students be y_2.
$e(4, 4, 4)$. Let the number of such students be z.
1) $|J_2| = |J_3|$,
only students using J_2 are $e(2, 4, 5)$ – they use one share each,
only students using J_3 are $e(3, 4, 5)$ – they use one share each.
Hence $y_1 = y_2$. We call them both y.

$f(31,19) \le \frac{54}{133}$

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$e(2,4,5)$. Let the number of such students be y_1
$e(3,4,5)$. Let the number of such students be y_2.
$e(4,4,4)$. Let the number of such students be z.
1) $|J_2| = |J_3|$,
only students using J_2 are $e(2,4,5)$ – they use one share each,
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$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$.
Contradiction.

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For $f(67,21)$ it fails!

The Train Method

We developed the Train Method which showed settled $f(67, 21)$ and 13 other problems we could not do.

Upshot

Let

$$A = \{(m, s) \mid 2 \le s \le 100 \text{ and } s < m \le 110 \text{ and } m, s \text{ rel prime}\}$$

There are 3520 pairs (m, s) in A. We solved **all** of them!

- For 2301 of them $f(m, s) = \mathrm{FC}(m, s)$. That is $\sim 65.37\%$.
- For 329 of them $f(m, s) = \mathrm{HALF}(m, s)$. That is $\sim 9.35\%$.
- For 186 of them $f(m, s) = \mathrm{INT}(m, s)$. That is $\sim 5.28\%$.
- For 111 of them $f(m, s) = \mathrm{MID}(m, s)$. That is $\sim 3.15\%$.
- For 240 of them $f(m, s) = \mathrm{EBM}(m, s)$. That is $\sim 6.28\%$.
- For 89 of them $f(m, s) = \mathrm{HBM}(m, s)$. That is $\sim 2.53\%$.
- For 250 of them $f(m, s) = \mathrm{GAP}(m, s)$. That is $\sim 7.10\%$.
- For 13 of them $f(m, s) = \mathrm{TRAIN}(m, s)$. That is $\sim 0.40\%$

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No. Did not work on

- $f(205, 178)$
- $f(226, 135)$
- $f(233, 141)$

The Scott Huddleston Technique

Scott Huddleston has an algorithm that is REALLY FAST and seems to ALWAYS WORK. Erik and Jacob understand it, nobody else does. They have replicated his results and think that YES it solves ALL problems.

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Richard Chatwin independently came up with the same algorithm and proved it worked, but never coded it up. He tells me its poly time and I believe this can be proved, though its not in his paper. His paper is here: `https://arxiv.org/abs/1907.08726`

Lessons Learned

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You never know where the next big project will come from!