On well-quasi-ordering finite trees

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Abstract. A new and simple proof is given of the known theorem that, if $T_1, T_2, \ldots$ is an infinite sequence of finite trees, then there exist i and j such that $i < j$ and T_i is homeomorphic to a subtree of T_j.

A *quasi-ordered set* is a set Q on which a reflexive and transitive relation $\leqslant$ is defined. Q and Q' will denote quasi-ordered sets. An infinite sequence $q_1, q_2, \ldots$ of elements of Q will be called *good* if there exist positive integers i, j such that $i < j$ and $q_i \leqslant q_j$; if not, the sequence will be called *bad*. A quasi-ordered set Q is *well-quasi-ordered* (*wqo*) if every infinite sequence of elements of Q is good.

A *graph* G consists (for our purposes) of a **finite** set $V(G)$ of elements called *vertices* of G and a subset $E(G)$ of the Cartesian product $V(G) \times V(G)$. The elements of $E(G)$ are called *edges* of G. If $(\xi, \eta) \in E(G)$, we call η a *successor* of ξ. If $\xi, \eta \in V(G)$, a *$\xi\eta$-path* is a sequence $\xi_0, \ldots, \xi_n$ of vertices of G such that $\xi_0 = \xi$, $\xi_n = \eta$ and $(\xi_{i-1}, \xi_i) \in E(G)$ for $i = 1, \ldots, n$. The sequence with sole term ξ is accepted as a $\xi\xi$-path. If there exists a $\xi\eta$-path, we say that *η follows ξ*. For the purposes of this paper, a *tree* is a graph T possessing a vertex $\rho(T)$ (called its *root*) such that, for every $\xi \in V(T)$, there exists a unique $\rho(T)\,\xi$-path in T. The letter T (with or without dashes or suffixes) will always denote a tree. For the purposes of this paper, a *homeomorphism* of T into T' is a function $\phi\colon V(T) \rightarrow V(T')$ such that, for every $\xi \in V(T)$, the images under ϕ of the successors of ξ follow *distinct* successors of $\phi(\xi)$. The set of all trees will be quasi-ordered by the rule that $T \leqslant T'$ if and only if there exists a homeomorphism of T into T'. This paper presents a new and shorter proof of the following theorem of Kruskal (2).

Theorem 1. *The set of all trees is wqo.*

If A, B are subsets of Q, a mapping $f\colon A \rightarrow B$ is *non-descending* if $a \leqslant f(a)$ for every $a \in A$. The class of finite subsets of Q will be denoted by SQ, and will be quasi-ordered by the rule that $A \leqslant B$ if and only if there exists a one-to-one non-descending mapping of A into B, where A, B denote members of SQ. The Cartesian product $Q \times Q'$ will be quasi-ordered by the rule that $(q_1, q_1') \leqslant (q_2, q_2')$ if and only if $q_1 \leqslant q_2$ and $q_1' \leqslant q_2'$. The cardinal number of a set A will be denoted by $|A|$.

The following two lemmas are well known (see (1)), but for the reader's convenience their proofs are given here.

Lemma 1. *If Q, Q' are wqo, then $Q \times Q'$ is wqo.*

Proof. We must prove an arbitrary infinite sequence $(q_1, q_1'), (q_2, q_2'), \ldots$ of elements of $Q \times Q'$ to be good. Call q_m *terminal* if there is no $n > m$ such that $q_m \leqslant q_n$. The number of q_m which are terminal must be finite, since otherwise they would form a bad subsequence of $q_1, q_2, \ldots$. Therefore there is an N such that q_r is not terminal if $r > N$. We can therefore select a positive integer $f(1) > N$, then an $f(2) > f(1)$ such that $q_{f(1)} \leqslant q_{f(2)}$, then an $f(3) > f(2)$ such that $q_{f(2)} \leqslant q_{f(3)}$ and so on. Since Q' is wqo, there exist i, j such that $i < j$ and $q'_{f(i)} \leqslant q'_{f(j)}$, whence $(q_{f(i)}, q'_{f(i)}) \leqslant (q_{f(j)}, q'_{f(j)})$ and therefore our original sequence is good.

LEMMA 2. *If Q is wqo, then SQ is wqo.*

Proof. Assume that the lemma is false. Select an $A_1 \in SQ$ such that A_1 is the first term of a bad sequence of members of SQ and $|A_1|$ is as small as possible. Then select an A_2 such that A_1, A_2 (in that order) are the first two terms of a bad sequence of members of SQ and $|A_2|$ is as small as possible. Then select an A_3 such that A_1, A_2, A_3 (in that order) are the first three terms of a bad sequence of members of SQ and $|A_3|$ is as small as possible. Assuming the Axiom of Choice, this process yields a bad sequence $A_1, A_2, A_3, \ldots$. Since this sequence is bad, no A_i is empty: therefore we can select an element a_i from each A_i. Let $B_i = A_i - \{a_i\}$. If there existed a bad sequence $B_{f(1)}$, $B_{f(2)}, \ldots$ such that $f(1) \leqslant f(i)$ for all i, the sequence

$$A_1,\ A_2,\ \ldots,\ A_{f(1)-1},\ B_{f(1)},\ B_{f(2)},\ \ldots$$

would be bad (since $A_i \leqslant B_j$ entails $A_i \leqslant A_j$ and is therefore impossible if $i < j$). Since this would contradict the definition of $A_{f(1)}$, there can be no bad sequence $B_{f(1)}, B_{f(2)}, \ldots$ such that $f(1) \leqslant f(i)$ for all i. It follows that the class ($\mathfrak{B}$, say) of sets B_i is wqo, since any bad sequence of sets B_i would have a (bad) infinite subsequence in which no suffix was less than the first. Therefore, by Lemma 1, $Q \times \mathfrak{B}$ is wqo. Therefore there exist i, j such that $i < j$ and $(a_i, B_i) \leqslant (a_j, B_j)$, which implies that $A_i \leqslant A_j$ and thus contradicts the badness of $A_1, A_2, \ldots$. This contradiction proves the lemma.

The *branch* of T at a vertex ξ is the tree R such that $V(R)$ is the set of those vertices of T which follow ξ and

$$E(R) = E(T) \cap (V(R) \times V(R)).$$

Proof of Theorem 1. Assume that the theorem is false. Select a tree T_1 such that T_1 is the first term of a bad sequence of trees and $|V(T_1)|$ is as small as possible. Then select a T_2 such that T_1, T_2 (in that order) are the first two terms of a bad sequence of trees and $|V(T_2)|$ is as small as possible. Continuing this process as in the proof of Lemma 2 yields a bad sequence $T_1, T_2, \ldots$. Let B_i be the set of branches of T_i at the successors of its root, and let $B = B_1 \cup B_2 \cup \ldots$. If there existed a bad sequence $R_1, R_2, \ldots$ such that $R_i \in B_{f(i)}$ and $f(1) \leqslant f(i)$ for every i, the sequence

$$T_1,\ T_2,\ \ldots,\ T_{f(1)-1},\ R_1,\ R_2,\ \ldots$$

would be bad (since $T_i \leqslant R \in B_j$ entails $T_i \leqslant T_j$ and is therefore impossible if $i < j$). Since this would contradict the definition of $T_{f(1)}$, there can be no bad sequence $R_1, R_2, \ldots$ such that $R_i \in B_{f(i)}$ and $f(1) \leqslant f(i)$ for every i. Since any bad sequence of elements of B would have a bad subsequence of this form, it follows that no sequence of elements of B is bad. Therefore B is wqo and hence, by Lemma 2, SB is wqo. Therefore

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$B_i \leqslant B_j$ for some pair i, j such that $i < j$. Therefore there is a one-to-one non-descending mapping ϕ: $B_i \rightarrow B_j$. For each $R \in B_i$, $R \leqslant \phi(R)$ and so there exists a homeomorphism h_R of R into $\phi(R)$. A homeomorphism h of T_i into T_j may thus be defined by writing $h(\rho(T_i)) = \rho(T_j)$ and making h coincide with h_R on the vertices of each $R \in B_i$. Therefore $T_i \leqslant T_j$, which contradicts the badness of T_1, T_2, ... and thus proves Theorem 1.

The Tree Theorem of (2) is stronger than Theorem 1 of the present paper, but the above proof of Theorem 1 can easily be adapted to prove the Tree Theorem by considering $X \times F(B)$ in place of SB (where X, F have the meanings stated in (2)). Because the necessary changes are easy to make, I have sacrificed this much generality in the interests of readability.

Note added 10 *August* 1963. It has been brought to the author's notice that Kruskal's proof of the Tree Theorem (2) anticipated a somewhat similar proof obtained independently by S. Tarkowski (*Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **8** (1960), 39–41).

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An application of harmonic coordinates in general relativity

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We suppose the first derivatives of the components of a continuous metric tensor to exhibit jumps across a non-null hypersurface. We shall show that harmonic coordinates (see Fock (1), p. 175) lead to an automatic smoothing of the metric.

Let x^m ($m = 1, 2, 3, 4$) be coordinates in a certain region of Riemannian space-time. We write $F \in (C^N, C^{N+K})$ to mean that $F(x^m)$ has continuous Nth partial derivatives throughout the region with its $(N+1)$th, ..., $(N+K)$th derivatives discontinuous only across a hypersurface $u(x^m) = 0$. Lichnerowicz ((2), p. 5) requires the metric tensor $g_{ij}(x^m)$ in 'admissible' coordinates to satisfy $g_{ij} \in (C^1, C^3)$ with respect to hypersurfaces for which $u \in (C^2, C^4)$. This state of affairs is preserved by a (C^2, C^4) coordinate transformation. Now suppose we leave the class of admissible coordinate systems by

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