Primitive Recursive Function and Ramsey Theory

Exposition by William Gasarch-U of MD

Bounds on a-ary Ramsey Numbers

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We need a way to express very fast growing functions.

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4. $g_1(x_1, \ldots, x_k)$, …, $g_n(x_1, \ldots, x_k)$,$h(x_1, \ldots, x_n)$ PR $\implies$

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5. $h(x_1, \ldots, x_{n+1})$ and $g(x_1, \ldots, x_{n-1})$ PR $\implies$

$$\begin{aligned} f(x_1, \ldots, x_{n-1}, 0) &= g(x_1, \ldots, x_{n-1}) \\ f(x_1, \ldots, x_{n-1}, m+1) &= h(x_1, \ldots, x_{n-1}, m, f(x_1, \ldots, x_{n-1}, m)) \end{aligned}$$

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The PR functions can be put in a hierarchy depending on how many times the recursion rule is used to build up to the function.

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What should we call this? Discuss

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f_6 and beyond have no name.

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Note One can show that any finite number of exponentials is in PR_3.

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$R_2(k) \le 2^{2k} = f_3(O(k))$. Level 3.
$R_3(k) \le \mathrm{TOW}(2k) = f_4(O(k))$. Level 4.
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- Is the function $f(a,k) = R_a(k)$ PR?
- Is $\mathrm{LR}(k)$ PR? If so then what level?

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Yes. We will see a contrived one on the next slide.

A Contrived Not PR Function

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Let $f_1, f_2, \ldots$ be all of the PR functions.

$$F(x) = f_x(x) + 1$$

is computable but not a PR function.

A "Natural" non PR Function

Def **Ackerman's function** is the function defined by

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\begin{aligned}
A(0, y) &= y + 1 \\
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1. A is obviously computable.
2. A grows faster than any PR function.
3. Since A is defined using a recursion which involves applying the function to itself there is no obvious way to take the definition and make it PR. Not a proof, an intuition.

Ackerman's Function is Natural: Security

https://ackerman-security-systems.pissedconsumer.com/customer-service.html

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They are called Ackerman Security since they claim that Burglar would have to be Ackerman(n)-good to break in.

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- There is a DS for this problem that can do n operations in $nA^{-1}(n)$ steps.
- One can show that there is no better DS.

So $nA^{-1}(n, n)$ is the exact upper and lower bound!

Ackerman's Function and Goodstein Seq

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This is called **Hereditary Base *n* Notation**

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Repeat the process:
Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, $\cdots$.

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Repeat the process:
Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, $\cdots$.
Vote Does the sequence:

- Goto infinity (and if so how fast- perhaps Ack-like?)
- Eventually stabilizes (e.g., goes to 18 and then stops there)
- Cycles- goes UP then DOWN then UP then DOWN ....

Ackerman's Function and Goodstein Seq

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Replace all of the 2's with 3's:

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0}$$

This number just went WAY up. Now subtract 1.

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The sequence goes to 0.
The number of steps for n to goto 0 is roughly $ACK(n, n)$.

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   YES, NO, UNKNOWN

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   NO. See next slide.

What is known about $\mathrm{LR}_a(k)$?

Thm For all a, k there exists $n = \mathrm{LR}_a(k)$ such that for all $\mathrm{COL}\colon \binom{\{k,\ldots,k+n\}}{a} \rightarrow [2]$ $\exists$ a large homog set.

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   For all $\alpha < \epsilon_0$, $f(a,k)$ is not in any PR_α.

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Num of colors matters—1st time in this course!

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4. Paris & Harrington(1977) showed LR could not be proven in PA, using Model Theory. Solovay & Ketonen (1981) showed LR not provable in PA via $f(a, k)$ growing fast.

Vote Is the LR Theorem a natural theorem? YES, NO, UNKNOWN TO SCIENCE.
Commentary on next slide.

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1. When did the Large Ramsey Theorem first appear?
   In Paris-Harrington paper that showed LR was Ind of PA.
   Thats an argument for LR being contrived.
2. LR is far more interesting than Godel's Sentence.
3. The proof of LR is interesting since you get it from infinite Ramsey but can't get it a more normal way.