# Small Ramsey Numbers

**Exposition by William Gasarch**

March 5, 2022

# Lets Party Like Its January of 2019

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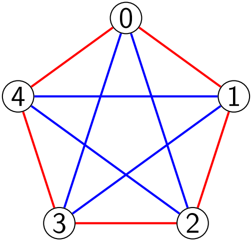
*For all 2-coloring of the edges of $K_6$ there is a mono $K_3$.*

**Question** What if we color the edges of $K_5$?

# Coloring of $K_5$ with no Mono $K_3$




This graph is not arbitrary.
$SQ_5 = \{x^2 \pmod 5 : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in SQ_5$ then RED.
- If $i - j \notin SQ_5$ then BLUE.

# Asymmetric Ramsey Numbers

**Definition** $R(a, b)$ is least $n$ such that for all 2-colorings of $K_n$ there is **either** a red $K_a$ or a blue $K_b$.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
3. $R(a, 2) = a$

# $R(a,b) \le R(a-1,b) + R(a,b-1)$

**Theorem** $R(a,b) \le R(a-1,b) + R(a,b-1)$
**Proof**
Let $n = R(a-1,b) + R(a,b-1)$. $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$.
**Case 1** $(\exists v)[\deg_R(v) \ge R(a-1,b)]$. Look at the $R(a-1,b)$ vertices that are RED to $v$. By Definition of $R(a-1,b)$ either

- There is a RED $K_{a-1}$. Combine with $v$ to get RED $K_a$.
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**Case 3**
$(\forall v)[\deg_R(v) \leq R(a-1, b) - 1 \wedge \deg_B(v) \leq R(a, b-1) - 1]$
$(\forall v)[\deg(v) \leq R(a-1, b) + R(a, b-1) - 2 = n - 2]$
Not possible since every vertex of $K_n$ has degree $n - 1$.

# Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
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# $R(3,4) \leq 9$

**Theorem** $R(3,4) \leq 9$.
Let $COL$ be a 2-coloring of the edges of $K_9$.
**Case 1** $(\exists v)[\deg_R(v) \geq 4]$. $v_1, v_2, v_3, v_4$ are RED to $v$.

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**Case 2** $(\exists v)[\deg_B(v) \geq 6]$. $v_1, v_2, v_3, v_4, v_5, v_6$ are BLUE to $v$.
Either:
(1) a RED $K_3$, or
(2) a BLUE $K_3$, which together with $v$ is a BLUE $K_4$.
**NOTE** Can't have any $\deg_R(v) \leq 2$.

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**NOTE** Can't have any $\deg_R(v) \leq 2$.

**Case 3** $(\forall v)[\deg_R(v) = 3]$. The RED subgraph has 9 nodes each of degree 3. Impossible!

# Reminder of the Odd Vertex Things

**Lemma** Let $G = (V, E)$ be a graph.

$$V_{\text{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\text{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

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Recall that for any graph $G = (V, E)$:

$$\sum_{v \in V_{\mathrm{even}}} \deg(v) + \sum_{v \in V_{\mathrm{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

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$$\sum_{v \in V_{\text{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\text{odd}}| \equiv 0 \pmod 2$.

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**Key:** $R(2,4)$ and $R(3,3)$ were both **even!**

**Theorem** $R(a,b) \le$

1. $R(a,b-1) + R(a-1,b)$ always.
2. $R(a,b-1) + R(a-1,b) - 1$ if $R(a,b-1) \equiv R(a-1,b) \equiv 0 \pmod 2$

# Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 - 1 = 31.$
- $R(5,5) \leq R(4,5) + R(5,4) = 62.$

Are these tight?

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$COL(a,b) =$ RED if $a - b \equiv SQ \pmod 5$, BLUE OW.

**Note** $-1 = 2^2 \pmod 5$. Hence $a - b \in SQ$ iff $b - a \in SQ$. So the coloring is well defined.

# $R(3,3) \geq 6$

$COL(a,b) =$ RED if $a - b \equiv SQ \pmod 5$, BLUE OW.

- Squares mod 5: 1,4.
- If there is a RED triangle then $a-b$, $b-c$, $c-a$ all SQ's. SUM is 0. So

$$x^2 + y^2 + z^2 \equiv 0 \pmod 5 \text{ Can show impossible}$$

- If there is a BLUE triangle then $a-b$, $b-c$, $c-a$ all non-SQ's. Product of nonsq's is a sq. So $2(a-b)$, $2(b-c)$, $2(c-a)$ all squares. SUM to zero- same proof.

**UPSHOT** $R(3,3) = 6$ and the coloring used math of interest!

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Vertices are $\{0, \ldots, 16\}$.

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Same idea as above for $K_5$, but more cases.
**UPSHOT** $R(4,4) = 18$ and the coloring used math of interest!

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THATS IT.

No other $R(a,b)$ are known using NICE methods.

$R(5,5)$– I will give you a paper to read on that soon.

# Revisit those Numbers

Int means Interesting Math. Bor means Boring Math.

- $R(3,3) \le 6$. TIGHT. Int
- $R(3,4) \le 9$. TIGHT. Int
- $R(3,5) \le 14$. TIGHT. Int
- $R(3,6) \le 19$. KNOWN: 18. Upper Bd Bor, Lower Bd Int
- $R(3,7) \le 26$. KNOWN: 23. Upper Bd Bor, Lower Bd Int
- $R(4,4) \le 18$. TIGHT. Int
- $R(4,5) \le 31$. KNOWN: 25. Both bd Bor
- $R(5,5) \le 62$. KNOWN: Will see it in the paper I give out.

# Moral of the Story

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   *(Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.*

2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.