The Square Theorem

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The Square Theorem

Definition Let $G \in \mathbb{N}$ and $c \in N$. Let $\mathrm{COL}\colon [G] \times [G] \to [c]$.

1. A **mono** ***L*** is 3 points

$$(x, y), (x + d, y), (x, y + d)$$

   that are all the same color $(d \geq 1)$. (This should be called an *mono isosceles right triangle* but we just call it a *mono L*.)

2. A **mono Square** is 4 points

$$(x, y), (x + d, y), (x, y + d), (x + d, y + d)$$

   that are all the same color $(d \geq 1)$. This is a square.

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3. To prove **The Square Theorem** (about 2-coloring) we need to know that $GG(c)$ exists for a very large c.
4. More Colors: *For all c there exists $G = G(c)$ such that for all $\mathrm{COL}\colon [G] \times [G] \to [c]$ there exists a mono square.* Proof needs a larger c' for $GG(c')$.

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This is very typical of VDW-Ramsey Theory: a 2-coloring of BLAH is viewed as a X-coloring of a different object where X is quite large.

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- Easier to prove it from the Hales-Jewitt Theorem, which we won't be doing.

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Go to Whiteboard for rest of proof.