Schur’s Thm + FLT(for $n = 4$) implies Primes Infinite

July 19, 2023
Credit Where Credit is Due

The following people have used Ramsey Theory to show Primes are Infinite.

1. Alpoge (2015) used (1) Elementary NT, (2) VDW.
2. Granville (2017) used (1) Intermediary NT, (2) VDW.
3. Elshotz (2021) & Gasarch (2023) used (1) Intermediary NT, (2) Schur's Theorem.

1. Granville and Gasarch build on work from Alpoge.
2. Gasarch uses easier Ramsey Theory than the other two.
3. All three of these proofs are harder than the usual proof.
The following people have used Ramsey Theory to show Primes are Infinite.

1. Alpoge (2015) used (1) Elementary NT, (2) VDW.
2. Granville (2017) used (1) Intermediary NT, (2) VDW.
3. Elsholtz (2021) & Gasarch (2023) used (1) Intermediary NT, (2) Schur’s Theorem.

1. Granville and Gasarch build on work from Alpoge.
2. Gasarch uses easier Ramsey Theory than the other two.
3. All three of these proofs are harder than the usual proof.
Credit Where Credit is Due

The following people have used Ramsey Theory to show Primes are Infinite.

1. Alpoge (2015) used (1) Elementary NT, (2) VDW.
2. Granville (2017) used (1) Intermediary NT, (2) VDW.

Granville and Gasarch build on work from Alpoge. Gasarch uses easier Ramsey Theory than the other two. All three of these proofs are harder than the usual proof.
The following people have used Ramsey Theory to show Primes are Infinite.

1. Alpoge (2015) used (1) Elementary NT, (2) VDW.
2. Granville (2017) used (1) Intermediary NT, (2) VDW.
3. Elshotz (2021) & Gasarch (2023) used (1) Intermediary NT, (2) Schur’s Theorem.
Credit Where Credit is Due

The following people have used Ramsey Theory to show Primes are Infinite.

1. Alpoge (2015) used (1) Elementary NT, (2) VDW.
2. Granville (2017) used (1) Intermediary NT, (2) VDW.
3. Elshotz (2021) & Gasarch (2023) used (1) Intermediary NT, (2) Schur’s Theorem.

1. Granville and Gasarch build on work from Alpoge.
The following people have used Ramsey Theory to show Primes are Infinite.

1. Alpoge (2015) used (1) Elementary NT, (2) VDW.
2. Granville (2017) used (1) Intermediary NT, (2) VDW.
3. Elshotz (2021) & Gasarch (2023) used (1) Intermediary NT, (2) Schur’s Theorem.

1. Granville and Gasarch build on work from Alpoge.
2. Gasarch uses easier Ramsey Theory than the other two.
Credit Where Credit is Due

The following people have used Ramsey Theory to show Primes are Infinite.

1. Alpoge (2015) used (1) Elementary NT, (2) VDW.
2. Granville (2017) used (1) Intermediary NT, (2) VDW.
3. Elshotz (2021) & Gasarch (2023) used (1) Intermediary NT, (2) Schur’s Theorem.

1. Granville and Gasarch build on work from Alpoge.
2. Gasarch uses easier Ramsey Theory than the other two.
3. All three of these proofs are harder than the usual proof.
Background Needed

July 19, 2023
Schur’s Theorem

Thm $(\forall c)(\exists S = S(c))$ st for all $c$-colorings $\text{COL}: [S] \to [c]$ there exists $x, y, z$ monochromatic such that $x + y = z$. 

Proof We determine $S$ later. Given $\text{COL}$ we define $\text{COL}'$ as follows:

$\text{COL}'(x, y) = \text{COL}(|x - y|)$.

There exists a $\text{COL}'$-homog set $H$ of size 3 (that's all we need!). Say its $a < b < c$

$\text{COL}'(c, b) = \text{COL}'(b, a) = \text{COL}'(c, a)$

So $\text{COL}(c - b) = \text{COL}(b - a) = \text{COL}(c - a)$.

Let $x = c - b$, $y = b - a$, $z = c - a$. So let $S(c) = \mathbb{R}(3; c)$ (homog set 3, colors $c$).
Schur’s Theorem

**Thm** \((\forall c)(\exists S = S(c))\) st for all \(c\)-colorings \(\text{COL}: [S] \to [c]\) there exists \(x, y, z\) monochromatic such that \(x + y = z\).

**Pf** We determine \(S\) later. Given \(\text{COL}\) we define \(\text{COL}'([S]) \to [c]\) as follows:

\[
\text{COL}'(x, y) = \text{COL}(|x - y|).
\]
**Schur’s Theorem**

**Thm** \((\forall c)(\exists S = S(c))\) st for all \(c\)-colorings \(\text{COL}: [S] \to [c]\) there exists \(x, y, z\) monochromatic such that \(x + y = z\).

**Pf** We determine \(S\) later. Given \(\text{COL}\) we define \(\text{COL}'([S]\binom{2}{2}) \to [c]\) as follows:

\[
\text{COL}'(x, y) = \text{COL}(|x - y|).
\]

There exists a \(\text{COL}'\)-homog set \(H\) of size 3 (that's all we need!). Say its \(a < b < c\)

\[
\text{COL}'(c, b) = \text{COL}'(b, a) = \text{COL}'(c, a)
\]
Schur’s Theorem

**Thm** \((\forall c)(\exists S = S(c))\) st for all \(c\)-colorings \(\text{COL} : [S] \rightarrow [c]\) there exists \(x, y, z\) monochromatic such that \(x + y = z\).

**Pf** We determine \(S\) later. Given \(\text{COL}\) we define \(\text{COL}'([S]/2) \rightarrow [c]\) as follows:

\[
\text{COL}'(x, y) = \text{COL}(|x - y|).
\]

There exists a \(\text{COL}'\)-homog set \(H\) of size 3 (that's all we need!). Say its \(a < b < c\)

\[
\text{COL}'(c, b) = \text{COL}'(b, a) = \text{COL}'(c, a)
\]

So

\[
\text{COL}(c - b) = \text{COL}(b - a) = \text{COL}(c - a)
\]
Schur’s Theorem

**Thm** \((\forall c)(\exists S = S(c))\) st for all \(c\)-colorings \(\text{COL} : [S] \rightarrow [c]\) there exists \(x, y, z\) monochromatic such that \(x + y = z\).

**Pf** We determine \(S\) later. Given \(\text{COL}\) we define \(\text{COL}' : [S] \rightarrow [c]\) as follows:

\[
\text{COL}'(x, y) = \text{COL}(|x - y|).
\]

There exists a \(\text{COL}'\)-homog set \(H\) of size 3 (thats all we need!). Say its \(a < b < c\)
\[
\text{COL}'(c, b) = \text{COL}'(b, a) = \text{COL}'(c, a)
\]
So
\[
\text{COL}(c - b) = \text{COL}(b - a) = \text{COL}(c - a)
\]
Let \(x = c - b, y = b - a, z = c - a\).
Schur’s Theorem

**Thm** \((\forall c)(\exists S = S(c))\) st for all \(c\)-colorings \(\text{COL} : [S] \to [c]\) there exists \(x, y, z\) monochromatic such that \(x + y = z\).

**Pf** We determine \(S\) later. Given \(\text{COL}\) we define \(\text{COL}'([S]/2) \to [c]\) as follows:

\[
\text{COL}'(x, y) = \text{COL}(|x - y|).
\]

There exists a \(\text{COL}'\)-homog set \(H\) of size 3 (thats all we need!). Say its \(a < b < c\)

\[
\text{COL}'(c, b) = \text{COL}'(b, a) = \text{COL}'(c, a)
\]

So

\[
\text{COL}(c - b) = \text{COL}(b - a) = \text{COL}(c - a)
\]

Let \(x = c - b, y = b - a, z = c - a\).

So let \(S(c) = R(3; c)\) (homog set 3, colors \(c\)).
Fermat’s Last Theorem

In 1637 Fermat wrote in the margins of *Arithmetica*, a book on Number Theory by Diophantus, the following (translated from Latin)

\[
(\forall n \geq 3)(\forall x, y, z \in \mathbb{N} - \{0\}) \quad x^n + y^n \neq z^n
\]

This has come to be known as Fermat’s Last Theorem.
Fermat’s Last Theorem

In 1637 Fermat wrote in the margins of *Arithmetica*, a book on Number Theory by Diophantus, the following (translated from Latin)

*To divide a cube into two cubes, a fourth power, or in general any power whatever above the second into two powers of the same denomination, is impossible, and I have assuredly found a proof of this, but the margin is too narrow to contain it.*
Fermat’s Last Theorem

In 1637 Fermat wrote in the margins of *Arithmetica*, a book on Number Theory by Diophantus, the following (translated from Latin)

*To divide a cube into two cubes, a fourth power, or in general any power whatever above the second into two powers of the same denomination, is impossible, and I have assuredly found a proof of this, but the margin is too narrow to contain it.*

In modern terminology:

\[(\forall n \geq 3)(\forall x, y, z \in \mathbb{N} - \{0\})[x^n + y^n \neq z^n].\]

This has come to be known as **Fermat’s Last Theorem**.
Did Fermat Have a Proof?

Arguments Against

1) He proved the $n=4$ case later in his life.
2) Andrew Wiles proved FLT in the early 1990s with techniques far beyond what Fermat could have known.

Arguments For

1) The 7th Dr. Who had a 5-line proof that uses Boolean Algebra.
2) The 11th Dr. Who gave The real proof to a group of geniuses to gain their trust. He later said that it was Fermat's original proof (possible but unlikely) but that Fermat didn't write it down since he died in a duel (not true). The writers of the show either confused Galois with Fermat or meant to say that Fermat died in a duel in a dual timeline.
Did Fermat Have a Proof?

Arguments Against

1) He proved the $n = 4$ case later in his life.
Did Fermat Have a Proof?

Arguments Against

1) He proved the $n = 4$ case later in his life.
2) Andrew Wiles proved FLT in the early 1990s with techniques far beyond what Fermat could have known.
Did Fermat Have a Proof?

Arguments Against

1) He proved the \( n = 4 \) case later in his life.
2) Andrew Wiles proved FLT in the early 1990s with techniques far beyond what Fermat could have known.

Arguments For
Did Fermat Have a Proof?

**Arguments Against**

1) He proved the $n = 4$ case later in his life.
2) Andrew Wiles proved FLT in the early 1990s with techniques far beyond what Fermat could have known.

**Arguments For**

1) The 7th Dr. Who had a 5-line proof that uses Boolean Algebra.
Did Fermat Have a Proof?

**Arguments Against**

1) He proved the $n = 4$ case later in his life.
2) Andrew Wiles proved FLT in the early 1990s with techniques far beyond what Fermat could have known.

**Arguments For**

1) The 7th Dr. Who had a 5-line proof that uses Boolean Algebra.
2) The 11th Dr. Who gave **The real proof** to a group of geniuses to gain their trust. He later said that it was Fermat’s original proof (possible but unlikely) but that Fermat didn’t write it down since he died in a duel (not true). The writers of the show either confused Galois with Fermat or meant to say that Fermat died in a duel in a dual timeline.
More Fiction about Fermat’s Last Theorem

In *Star Trek: TNG*, the episode *The Royale* which aired on March 27, 1989, Captain Picard, in the 24th Century is working on Fermat’s Last Theorem, which is still OPEN.

Whoops

In *Star Trek: DSN*, the episode *Facets* which aired on June 12, 1995, Dax says that one of her previous hosts, Tobin, had done the most creative work on Fermat’s Last Theorem since Wiles. My guess is that Tobin wrote this limerick:

A challenge for many long ages
Had baffled the savants and sages
Yet at last came the light
Seems that Fermat was right
To the margin add 200 pages.
More Fiction about Fermat’s Last Theorem

In *Star Trek: TNG*, the episode *The Royale* which aired on March 27, 1989, Captain Picard, in the 24th Century is working on Fermat’s Last Theorem, which is still OPEN. *Whoops*

In *Star Trek: DSN*, the episode *Facets* which aired on June 12, 1995, Dax says that one of her previous hosts, Tobin, had done the most creative work on Fermat’s Last Theorem since Wiles. My guess is that Tobin wrote this limerick:

*A challenge for many long ages
Had baffled the savants and sages
Yet at last came the light
Seems that Fermat was right
To the margin add 200 pages.*
In **Star Trek: TNG**, the episode **The Royale** which aired on March 27, 1989, Captain Picard, in the 24th Century is working on Fermat’s Last Theorem, which is still OPEN. **Whoops**

In **Star Trek: DSN**, the episode **Facets** which aired on June 12, 1995, Dax says that one of her previous hosts, Tobin, had done *the most creative work on Fermat’s Last Theorem since Wiles*. 

---

**A challenge for many long ages**
Had baffled the savants and sages
Yet at last came the light
Seems that Fermat was right
To the margin add 200 pages.
More Fiction about Fermat’s Last Theorem

In Star Trek: TNG, the episode The Royale which aired on March 27, 1989, Captain Picard, in the 24th Century is working on Fermat’s Last Theorem, which is still OPEN. Whoops

In Star Trek: DSN, the episode Facets which aired on June 12, 1995, Dax says that one of her previous hosts, Tobin, had done the most creative work on Fermat’s Last Theorem since Wiles.

My guess is that Tobin wrote this limerick:

A challenge for many long ages
Had baffled the savants and sages
Yet at last came the light
Seems that Fermat was right
To the margin add 200 pages.
Proof that Primes are Infinite

July 19, 2023
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

Let $p_1, \dots, p_L$ be the following coloring:

$$\text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \mod 4, \ldots, a_L \mod 4)$$

By Schur's Theorem there exists $x, y, z$ of the same color with $x + y = z$.

Assume the color is $(e_1, \ldots, e_L)$.

$$x = p_4 x_1^{e_1} \cdots p_4 x_L^{e_L} + e_1^{e_1} \cdots e_L^{e_L}$$
$$y = p_4 y_1^{e_1} \cdots p_4 y_L^{e_L} + e_1^{e_1} \cdots e_L^{e_L}$$
$$z = p_4 z_1^{e_1} \cdots p_4 z_L^{e_L} + e_1^{e_1} \cdots e_L^{e_L}$$

This violates FLT for $n = 4$. 

Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**

Assume, BWOC, that the primes are finite. 

Let 

\[ \text{COL} : \mathbb{N} \to \{0, 1, 2, 3\} \]

be the following coloring:

\[ \text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \text{mod } 4, \ldots, a_L \text{mod } 4) \]

By Schur's Thm there exists \( x, y, z \) same color with 

\[ x + y = z \]

Assume the color is \((e_1, \ldots, e_L)\).

\[ x = p_4^{e_1} \cdots p_L^{e_L} \]

\[ y = p_4^{e_1} \cdots p_L^{e_L} \]

\[ z = p_4^{e_1} \cdots p_L^{e_L} \]

This violates FLT for \( n = 4 \).
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**

Assume, BWOC, that the primes are finite. $p_1, \ldots, p_L$. 

Let $\text{COL} : \mathbb{N} \rightarrow \{0, 1, 2, 3\}$ be the following coloring:

$$\text{COL}(p_1^a_1 \cdots p_L^a_L) = (a_1 \text{mod } 4, \ldots, a_L \text{mod } 4)$$

By Schur's Thm there exists $x, y, z$ same color with $x + y = z$. Assume the color is $(e_1, \ldots, e_L)$. 

$$x = p_4^{e_1} \cdots p_L^{e_L} + e_1 \cdots p_L^{e_L} + p_4^{e_1} \cdots p_L^{e_L} + e_1 \cdots p_L^{e_L} = p_4^{e_1} \cdots p_L^{e_L} + p_4^{e_1} \cdots p_L^{e_L} + p_4^{e_1} \cdots p_L^{e_L} + p_4^{e_1} \cdots p_L^{e_L}$$

This violates FLT for $n = 4$. 
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**
Assume, BWOC, that the primes are finite. $p_1, \ldots, p_L$.
Let $\text{COL}: \mathbb{N} \rightarrow \{0, 1, 2, 3\}^L$ be the following coloring:
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**
Assume, BWOC, that the primes are finite. $p_1, \ldots, p_L$.
Let $COL : \mathbb{N} \to \{0, 1, 2, 3\}^L$ be the following coloring:

$$COL(p_1^{a_1} \cdot \cdot \cdot p_L^{a_L}) = (a_1 \pmod{4}, \ldots, a_L \pmod{4})$$
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**

Assume, BWOC, that the primes are finite. \( p_1, \ldots, p_L \).

Let \( \text{COL}: \mathbb{N} \to \{0, 1, 2, 3\}^L \) be the following coloring:

\[
\text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod{4}, \ldots, a_L \pmod{4})
\]

By Schur’s Thm there exists \( x, y, z \) same color with \( x + y = z \).
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**
Assume, BWOC, that the primes are finite. $p_1, \ldots, p_L$.
Let $\text{COL}: \mathbb{N} \rightarrow \{0, 1, 2, 3\}^L$ be the following coloring:

$$\text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod{4}, \ldots, a_L \pmod{4})$$

By Schur’s Thm there exists $x, y, z$ same color with $x + y = z$.
Assume the color is $(e_1, \ldots, e_L)$. 

This violates FLT for $n = 4$. 

Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**
Assume, BWOC, that the primes are finite. $p_1, \ldots, p_L$.

Let $\text{COL} : \mathbb{N} \to \{0, 1, 2, 3\}^L$ be the following coloring:

$$\text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod{4}, \ldots, a_L \pmod{4})$$

By Schur’s Thm there exists $x, y, z$ same color with $x + y = z$.
Assume the color is $(e_1, \ldots, e_L)$.

\begin{align*}
x &= p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L} \\
y &= p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L} \\
z &= p_1^{4z_1+e_1} \cdots p_L^{4z_n+e_L}
\end{align*}

This violates FLT for $n = 4$. 
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**
Assume, BWOC, that the primes are finite. $p_1, \ldots, p_L$.
Let $COL : \mathbb{N} \rightarrow \{0, 1, 2, 3\}^L$ be the following coloring:

$$COL(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod{4}, \ldots, a_L \pmod{4})$$

By Schur’s Thm there exists $x, y, z$ same color with $x + y = z$.
Assume the color is $(e_1, \ldots, e_L)$.

$$x = p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L}$$
$$y = p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L}$$
$$z = p_1^{4z_1+e_1} \cdots p_L^{4z_n+e_L}$$

$x + y = z$
**Proof that Primes are Infinite**

**Thm** The number of primes is infinite.

**Pf**

Assume, BWOC, that the primes are finite. $p_1, \ldots, p_L$.

Let $\text{COL}: \mathbb{N} \rightarrow \{0, 1, 2, 3\}^L$ be the following coloring:

$$\text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod 4, \ldots, a_L \pmod 4)$$

By Schur’s Thm there exists $x, y, z$ same color with $x + y = z$.

Assume the color is $(e_1, \ldots, e_L)$.

\[
\begin{align*}
x &= p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L} \\
y &= p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L} \\
z &= p_1^{4z_1+e_1} \cdots p_L^{4z_n+e_L}
\end{align*}
\]

$x + y = z$

\[
\begin{align*}
p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L} + p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L} &= p_1^{4z_1+e_1} \cdots p_L^{4z_n+e_L}
\end{align*}
\]

This violates FLT for $n = 4$. 
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**

Assume, BWOC, that the primes are finite. \( p_1, \ldots, p_L \).

Let \( \text{COL}: \mathbb{N} \to \{0, 1, 2, 3\}^L \) be the following coloring:

\[
\text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod{4}, \ldots, a_L \pmod{4})
\]

By Schur's Thm there exists \( x, y, z \) same color with \( x + y = z \).

Assume the color is \((e_1, \ldots, e_L)\).

\[
x = p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L}
\]
\[
y = p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L}
\]
\[
z = p_1^{4z_1+e_1} \cdots p_L^{4z_n+e_L}
\]

\[
x + y = z
\]

\[
(p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L} + p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L}) = p_1^{4z_1+e_1} \cdots p_L^{4z_n+e_L}
\]

\[
p_1^{4x_1} \cdots p_L^{4x_L} + p_1^{4y_1} \cdots p_L^{4y_L} = p_1^{4z_1} \cdots p_L^{4z_n}
\]

This violates FLT for \( n = 4 \).
Proof that Primes are Infinite

**Thm** The number of primes is infinite.

**Pf**

Assume, BWOC, that the primes are finite. \( p_1, \ldots, p_L \).

Let \( \text{COL}: \mathbb{N} \rightarrow \{0, 1, 2, 3\}^L \) be the following coloring:

\[
\text{COL}(p_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod{4}, \ldots, a_L \pmod{4})
\]

By Schur’s Thm there exists \( x, y, z \) same color with \( x + y = z \).

Assume the color is \( (e_1, \ldots, e_L) \).

\[
\begin{align*}
x &= p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L} \\
y &= p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L} \\
z &= p_1^{4z_1+e_1} \cdots p_L^{4z_L+e_L}
\end{align*}
\]

\[
x + y = z \\
p_1^{4x_1+e_1} \cdots p_L^{4x_L+e_L} + p_1^{4y_1+e_1} \cdots p_L^{4y_L+e_L} = p_1^{4z_1+e_1} \cdots p_L^{4z_L+e_L}
\]

\[
\begin{align*}
p_1^{4x_1} \cdots p_L^{4x_L} + p_1^{4y_1} \cdots p_L^{4y_L} &= p_1^{4z_1} \cdots p_L^{4z_L} \\
(p_1^{x_1} \cdots p_L^{x_L})^4 + (p_1^{y_1} \cdots p_L^{y_L})^4 &= (p_1^{z_1} \cdots p_L^{z_L})^4
\end{align*}
\]

This violates FLT for \( n = 4 \).