TAKE HOME FINAL FOR CMSC 752
DUE DUE May 19
NO Dead Cat Day

Rules Same as the HW: You can get help but you must understand and hand in your own work. You can use ChatGPT if you want to hand in answers that are really bad.

There are six problems and they add up to 100 points.

1. (4 points)

   (a) What was your favorite part of the course (can be a theorem or a small area, like Hypergraph Ramsey Theorems)? Why that one? (Answer seriously)

   (b) What was your least favorite part of the course (can be a theorem or a small area, like well quasi orders)? Should I omit it next time I teach the course? (Answer seriously)

   (c) Give a funny answer to part a above. (e.g., I liked that Bill had three students named Danesh, with Danesh-1 being the real Danesh, though your answer can also be a math thing.)

   (d) Give a funny answer to part b above. (e.g., I hated that Bill had three students named Kelin, with Kelin-1 NOT beingt the real Kelin, though your answer can also be a math thing.)

2. (24 points) Find a constant A such that the following holds:

   For large n, for all COL: $\binom{[n]}{2} \rightarrow [2]$ *there exists at least* $\sim A\binom{n}{10}$ *monochromatic* K_{10}*'s.*

   Two caveats

   - The $\sim$ means you can ignore lower order terms. For example, if you determined that the number of mono K_{10}'s is at least $89\binom{n}{10} - \text{GEN}(10)n^9$ then $A = 89$ is the answer.

     FYI: GEN is the Gen number, which is ACK(10, 10).

     FYI: $89\binom{n}{10} - \text{GEN}(10)n^9$ is NOT the answer.

   - If you find that A is a Ramsey Number then use the upper or lower bounds on Ramsey theory that we know:

     $$2^{k/2} \leq R(k) \leq 2^{2k}$$

     to get a value of A.

     (I know that better upper and lower bounds are known but this is the level of precision that I care about.)

3. (14 points) The notation $\mathsf{N}^{\geq 1}$ means naturals that are ≥ 1.

   Prove the following.

   Let $c \in \mathsf{N}^{\geq 1}$. Let COL: $\mathsf{N}^{\geq 1} \rightarrow [c]$. Then there exists $x_1, \ldots, x_n \in \mathsf{N}^{\geq 1}$ such that the following occurs.

   - $\mathrm{COL}(x_1) = \cdots = \mathrm{COL}(x_n)$.
   - $x_1 + 2x_2 + 3x_3 - 6x_4 + 5x_5 + 4x_6 = 0$.

   You may use the Real EVDW (see slides on the Extended VDW theorem). You may NOT use Rado's theorem- we are asking to to prove Rado's theorem for this case.

4. (10 points) The notation $\mathsf{N}^{\geq 1}$ means naturals that are ≥ 1.

   Prove the following.

   There exists $c \in \mathsf{N}^{\geq 1}$ *and* COL: $\mathsf{N}^{\geq 1} \rightarrow [c]$ *such that there is NO mono solution to* $x_1 + 2x_2 + 3x_3 - 20x_4 + 5x_5 + 4x_6 = 0$

   You may NOT use Rado's theorem- we are asking to to prove Rado's theorem for this case.

5. (24 points) Prove the following.

Let $c \in \mathsf{N}$. For all $\mathrm{COL}\colon \mathsf{R}^2 \to [c]$ *there exists points* $p_1, p_2, p_3 \in \mathsf{R}^2$ *such that the following occurs.*

- $\mathrm{COL}(p_1) = \mathrm{COL}(p_2) = \mathrm{COL}(p_3)$.
- *The triangle with vertices* p_1, p_2, p_3 *has area 1.*

6. (24- 6 points each part)

For this problem we use the following definitions.

1) A finite set $H \subseteq \mathsf{N}$ is *Large* if $|H| \geq \min(H)$.

2) Let $X, Y \subseteq \mathsf{N}$ (either could be finite or infinite). If COL: $X \rightarrow Y$ then H is *homog* if all of the elements of H are the same color.

3) Let $X, Y \subseteq \mathsf{N}$ (either could be finite or infinite). If COL: $X \rightarrow Y$ then H is *rainbow* if all of the elements of H are different colors.

(a) Prove the following statement using the 2-ary Ramsey Theorem.

(1-dim Can Ramsey) For all COL: $\mathsf{N} \rightarrow [\omega]$ *either there is an infinite homog set or an infinite rainbow set.*

(b) Prove the following statement in such a way that you get prim rec bounds on n as a function of k.

(1-dim Large Ramsey theorem) For all k there exists n such that for all COL: $\{k, \ldots, n\} \rightarrow [2]$ *there exists a large homog set.*

(c) Prove the following statement

(1-dim Large Can Ramsey Theorem) For all k there exists n such that for all COL: $\{k, \ldots, n\} \rightarrow [\omega]$ *either there exists a large homog set or there exists a large rainbow set.*