There is no 3-coloring of 15×15

There is a Large R Set

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$|X| \geq 75$.

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For $1 \leq i \leq 15$
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We will show $L > 105$, hence some four elements of X form a rectangle.

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Clearly $M_{\mathrm{R}} \le M_{\mathrm{N}}$.

Hence, to show $M_{\mathrm{N}} \ge 106$, it suffices to show $M_{\mathrm{R}} \ge 106$.

Well Known Theorem

The Min of

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occurs when all of the x_is are equal.

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We take $x_i = 75/15 = 5$.

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$$\sum_{i=1}^{15} \frac{x_i(x_i-1)}{2} \ge \sum_{i=1}^{15} \frac{5\times 4}{2} = 15 \times 10 = 150 > 106.$$

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That will be a **R** rectangle. Contradiction.

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Speculation

150 is **much larger** than 106. Hence 15 is way to big. Lets try 11.

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R appears $\geq \lceil \frac{11 \times 11}{3} \rceil = 41$ times.

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Need

$$\sum_{i=1}^{11} \binom{x_i}{2} \geq \binom{11}{2} = 55.$$

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relative to the constraint $\sum_{i=1}^{11} x_i = 41$
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In Fact It is know that 10×10 **is** 3-colorable.