Finite Ramsey Theorem For 3-Hypergraph: Better Bounds

Exposition by William Gasarch

November 21, 2024

Credit Where Credit is Due

The main theorem in these slides, in fact not just the 3-ary case but also the a-ary case, appeared in

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Here is a link

`https://www.cs.umd.edu/users/gasarch/TOPICS/canramsey/ErdosRado2.pdf`

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We will do much better.

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Now what? Discuss.

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$|H_{s+1}| \ge \frac{1}{2^s}|H_s| = \frac{1}{2^s}\frac{1}{2^{s-1}}|H_{s-1}| = \frac{1}{2^{s+(s-1)}}|H_{s-1}|.$

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We will later see how big we need n to be.

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Hence H is homog for COL.

How Big Does N Have to Be. Part II

We need

$$k \le \log s/2$$

$$2k \le \log s$$

$s \ge 2^{2k}$
$n \ge 2^{s^2/2} \ge 2^{2^{4k}}$
SO we can take $n = 2^{2^{4k}}$.

Final Notes

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Why did it take 2009-1952$=$ 57 years to improve the bound? Discuss.