BILL, RECORD LECTURE!!!!

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Goodstein Sequences

Exposition by William Gasarch-U of MD

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We can even write the exponents that are not already powers of 2 as sums of powers of 2.

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This is called **Hereditary Base *n* Notation**

Ackermann's Function and Goodstein Seq

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Replace all of the 2's with 3's:

$$3^{3^{3^{3^0}+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^{3^0}+3^0} + 3^{3^3+3^{3^0}} + 3^{3^3+3^0} + 3^{3^{3^0}+3^0}$$

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This number just went WAY up. Now subtract 1.

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Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, $\cdots$.

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Vote Does the sequence:

- Goto infinity (and if so how fast- perhaps Ack-like?)
- Eventually stabilizes (e.g., goes to 18 and then stops there)
- Cycles- goes UP then DOWN then UP then DOWN ....

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Really? Really!

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Note that the right most digit is 0. That will happen ∞ often.

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Now its a 2-digit number and use induction.

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3) **Natural Mathemtical Statement** We'll take this to be a statement of intereting math content. We will soon disuss an unnatural math statement.

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Next Slide will indicate why am asking this.

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3. Harvey Friedman has done much research on this. Here is one of his theorems:
   `https://cpb-us-w2.wpmucdn.com/u.osu.edu/dist/1/1952/files/2014/01/FIiniteSeqInc062214a-v9w7q4.pdf`

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Answer on Next Slide.

Is $R_a(k)$ Prim Rec?

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We will show that $R_3(k) \leq 2^{2^{4k}}$.

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We will see that $R_a(k)$ is bounded by a stack $a - 1$ 2's.