BILL, RECORD LECTURE!!!!

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A Variant on $R(3) = 6$

Exposition by William Gasarch

April 1, 2025

Credit Where Credit Is Due

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The questions raised in these slides are due to Paul Erdös.

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Ronald Graham, and

Shen Lin.

A Variant of $R(3) = 6$

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Is there a graph G w/o a K_4-subgraph such that (*) holds?

Terminology

Def Let $G = (V, E)$ be a graph. $\mathrm{RAM}(G)$ means that
For all $\mathrm{COL}\colon E \rightarrow [2]$ there exists a 3-homog set.

Vote on $\exists G$ w/o K_6, $\mathrm{RAM}(G)$ Holds

Is there a graph G such that $\mathrm{RAM}(G)$ and K_6 is NOT a subgraph.

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Answer on the next slide.

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Over $A(10, 10)$ vertices where A is Ackerman's function.

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(Graham) 8 vertices!

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(Shen) There is no such graph on 7 vertices. We skip this.

G Such That $\mathrm{RAM}(G)$, G Has No K_6 Subgraph, G Has 8 Vertices

G Such That RAM and No K_6

Let $G = (V, E)$ be the graph

G Such That RAM and No K_6

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$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

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Let $G = (V, E)$ be the graph

$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$E = \binom{V}{2} - \{(4, 5), (5, 6), (6, 7), (7, 8), (8, 4)\}$

Graham's Graph

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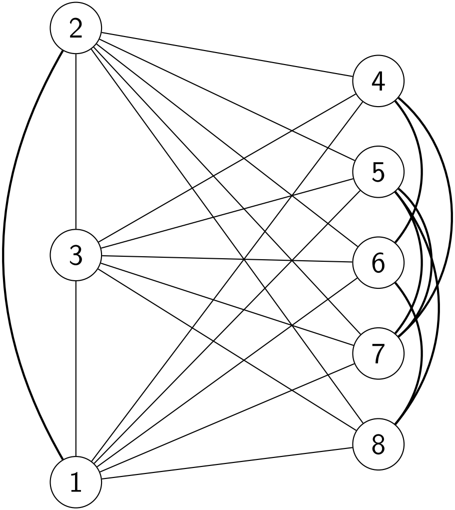

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Assume that $\exists$ $\mathrm{COL}\colon E \rightarrow [2]$ has no mono $\triangle$s.

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We assume $\mathrm{COL}(1, 2) =$ **B**, $\mathrm{COL}(1, 3) = \mathrm{COL}(2, 3) =$ **R**.

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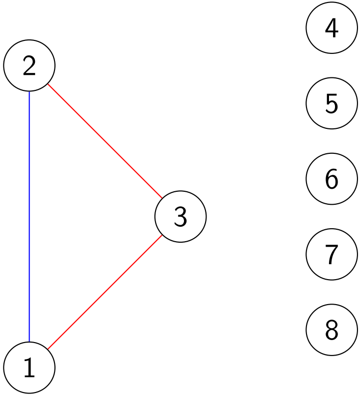

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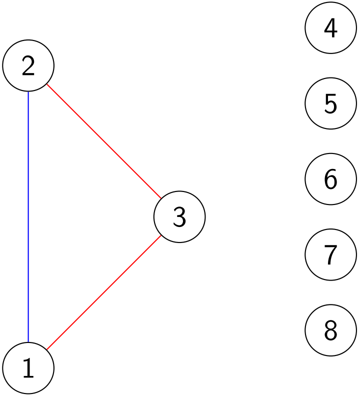

We show that, for all $4 \le i \le 8$, $\mathrm{COL}(3, i) =$ **B**.

$$\mathbf{COL}(3, i) = \mathsf{B}$$

$\mathrm{COL}(3, i) = \mathbf{B}$

Assume, BWOC, $\mathrm{COL}(3, 6) = \mathbf{R}$.

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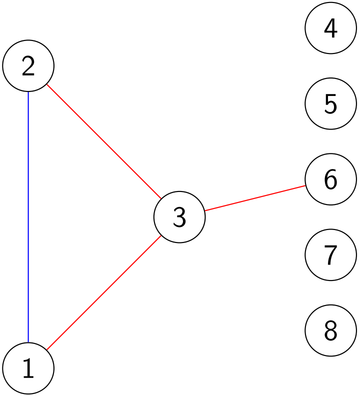

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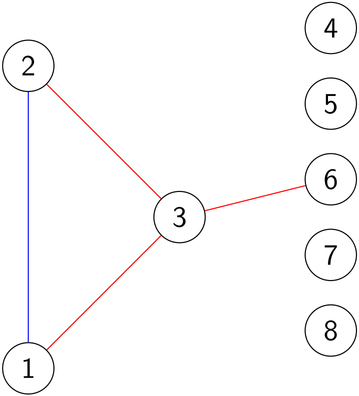

If $\mathrm{COL}(2,6) = \mathbf{R}$ then $2 - 3 - 6$ is $\mathbf{R}\triangle$. So $\mathrm{COL}(2,6) = \mathbf{B}$.

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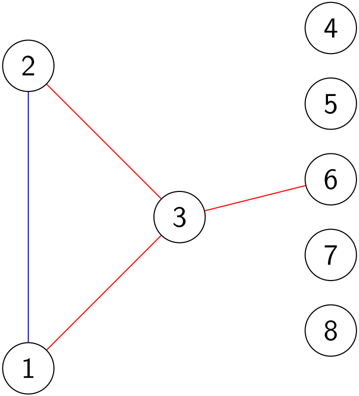

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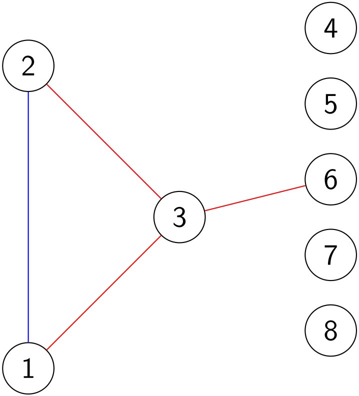

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If $\mathrm{COL}(1,6) = \mathbf{R}$ then $1 - 3 - 6$ is $\mathbf{R}\triangle$. So $\mathrm{COL}(1,6) = \mathbf{B}$.
So $1 - 2 - 6$ is a $\mathbf{B}\triangle$.

Lots of B Edges

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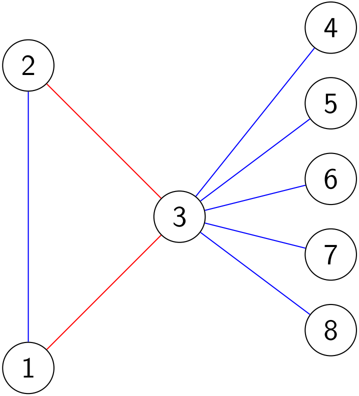

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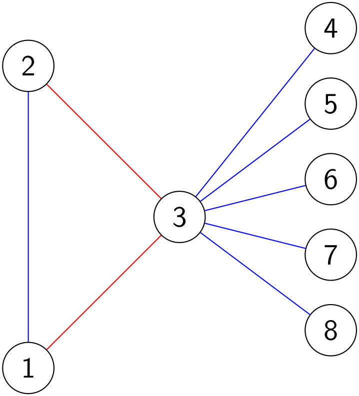
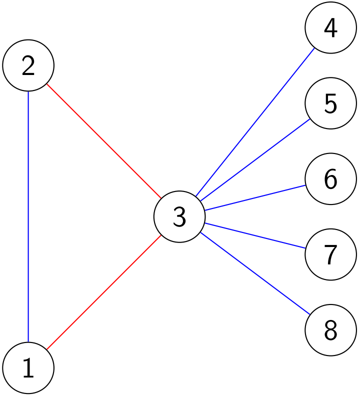


Recall that $(4,6),(4,7),(5,7),(5,8),(6,8)$ are edges of G

Lots of **B** Edges



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They must all be **R**.

Lots of R Edges

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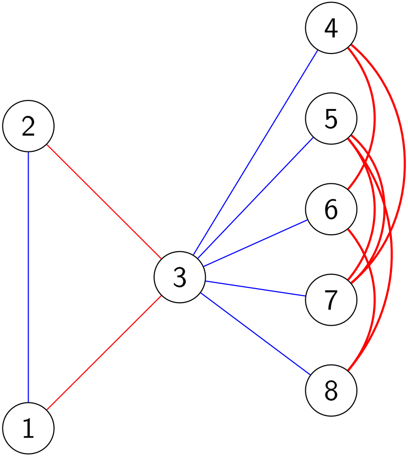
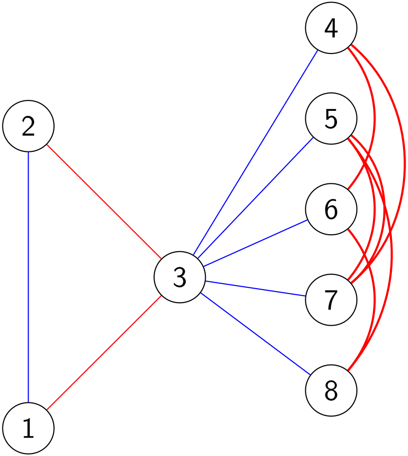
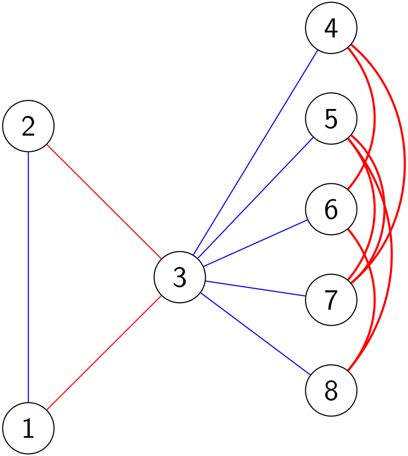


Lots of **R** Edges



Claim At most 2 of $(1, x)$ are **R**. Assume 3 are **R**.
Can assume $\mathrm{COL}(1, 4) =$ **R**.

Lots of R Edges



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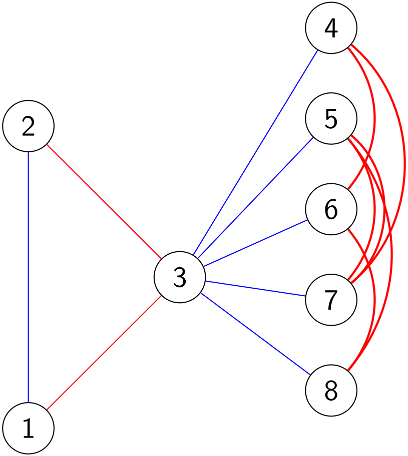

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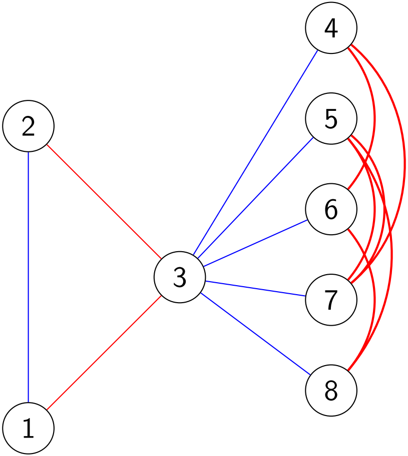

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If $\mathrm{COL}(1,7) =$ **R** then $1-4-7$ is $\triangle$.

Since 3 are **R** $\mathrm{COL}(1,5) = \mathrm{COL}(1,8) =$ **R**. So $1-5-8$ is $\triangle$.

Lots of R Edges: Recap and Extend

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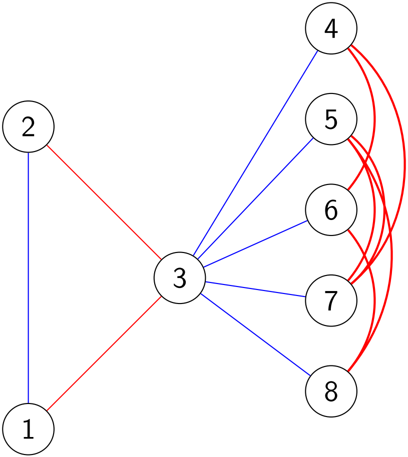

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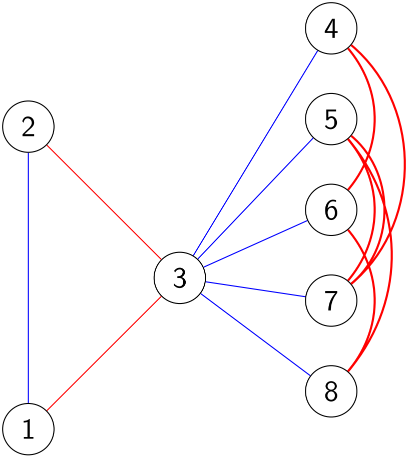

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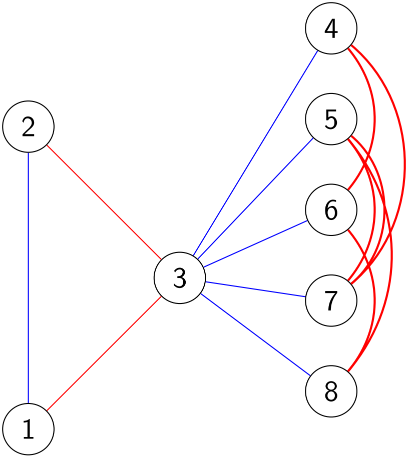

At most 2 of $(1, x)$ are **R**. **Sim** At most 2 of $(2, x)$ are **R**.

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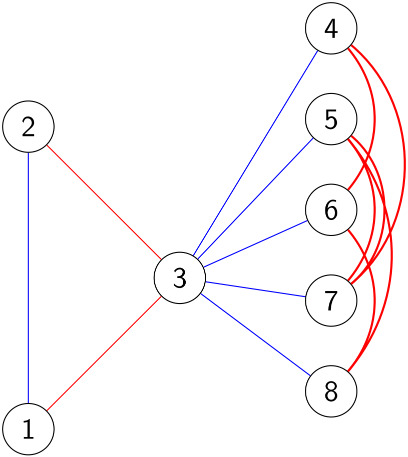

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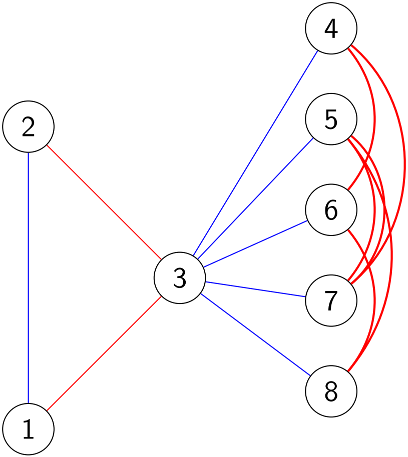

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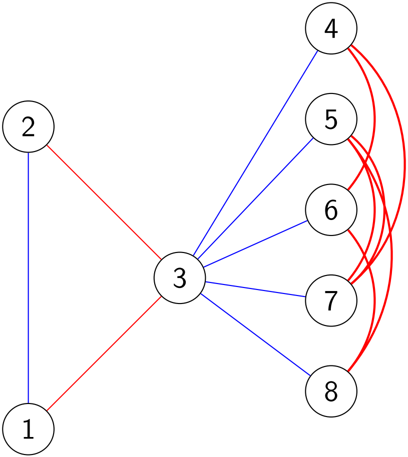

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Can assume $\mathrm{COL}(1, 6) = \mathrm{COL}(2, 6) =$ **B**.

Some More B Edges

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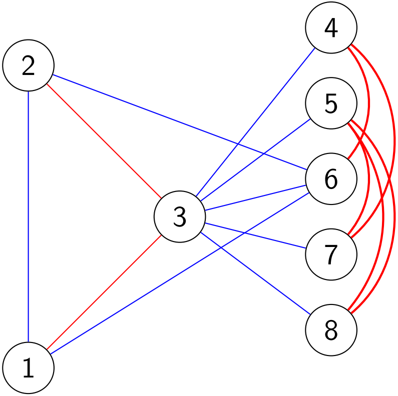

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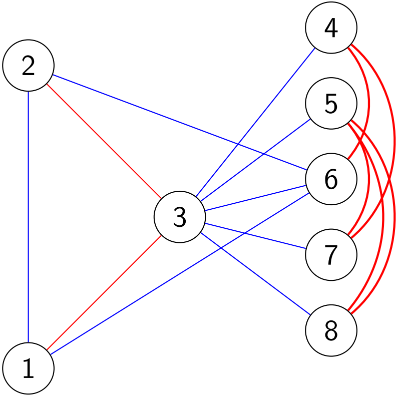

We have △: 1 − 2 − 6.

No G Such That

$\mathrm{RAM}(G)$,

G Has No K_6 Subgraph,

G Has 7 Vertices

We Skip But Its On The Slides

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Upshot We will skip this; however, you can read my slides if you are curious.

$G = (V, E)$. We Will 2-Color E w/No Mono $\triangle$s

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We do an example of a coloring of $G = (V, E)$ with no mono $\triangle$s.

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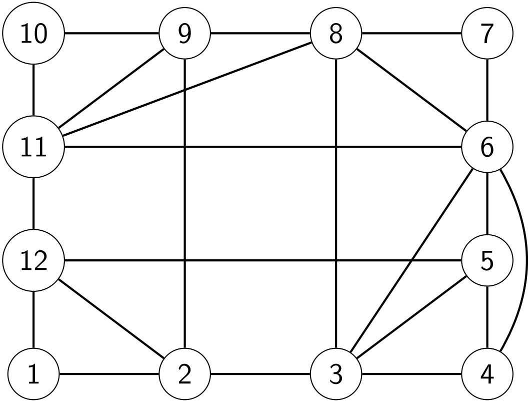

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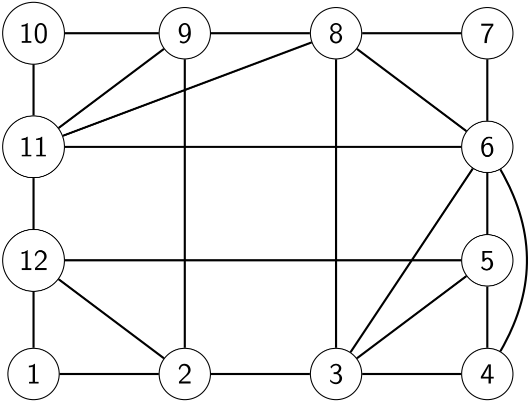

Partition. into Ind Sets:

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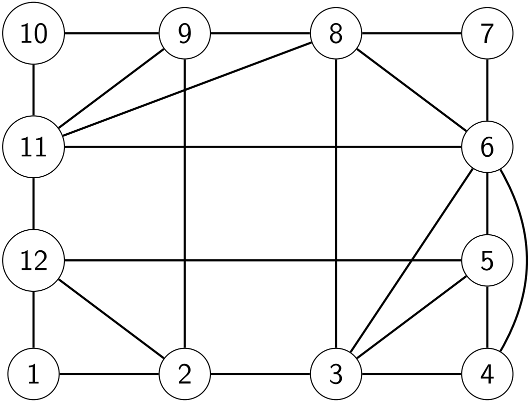
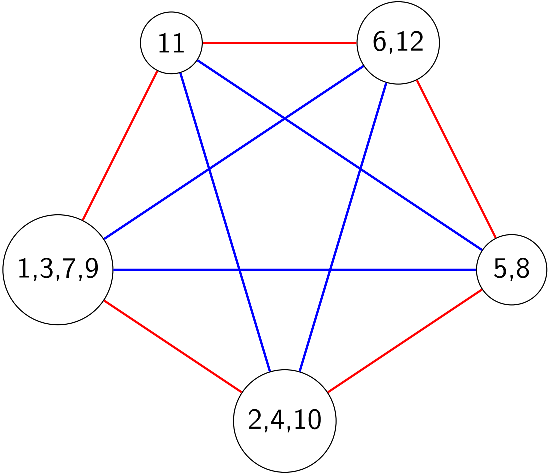


Partition. into Ind Sets:

$\{1, 3, 7, 9\}$, $\{2, 4, 10\}$, $\{5, 8\}$, $\{6, 12\}$, $\{11\}$.

Plan: View Ind Sets As Vertices of K_5



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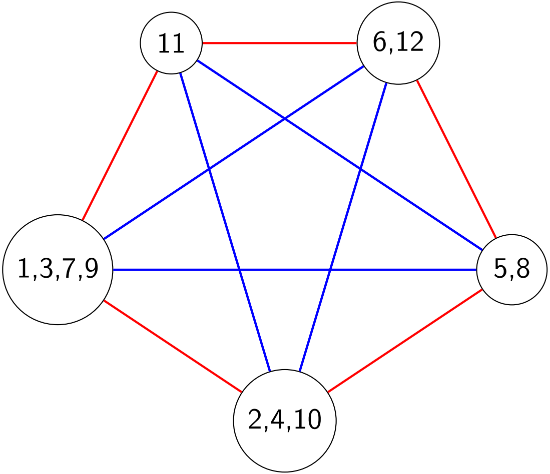

$\mathrm{COL}(i,j)$ is the color between the supernodes containing i,j.

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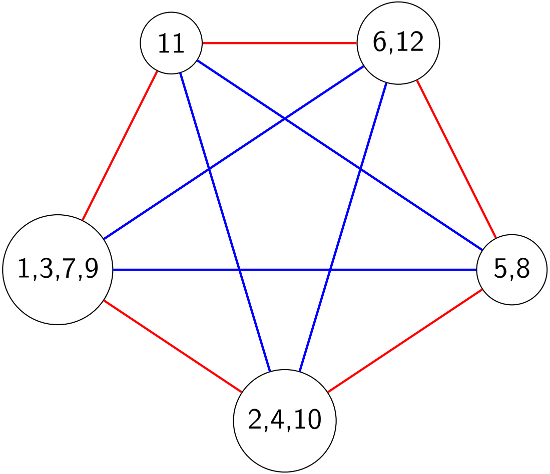

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Within a supernode there are no edges.

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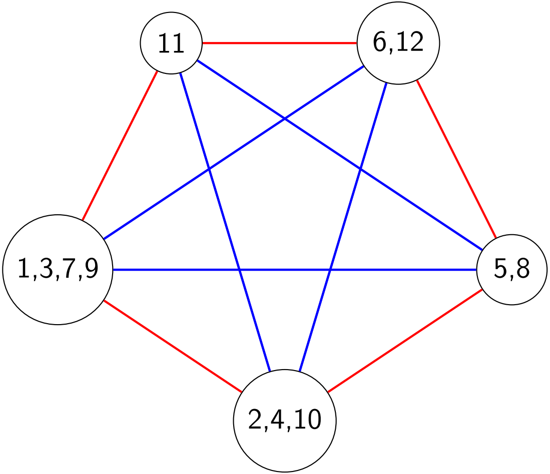

$\mathrm{COL}(i,j)$ is the color between the supernodes containing i,j.
Within a supernode there are no edges.
Easy to see there are no mono $\triangle$s.

The Coloring of The Edges of G w/No Mono $\triangle$s

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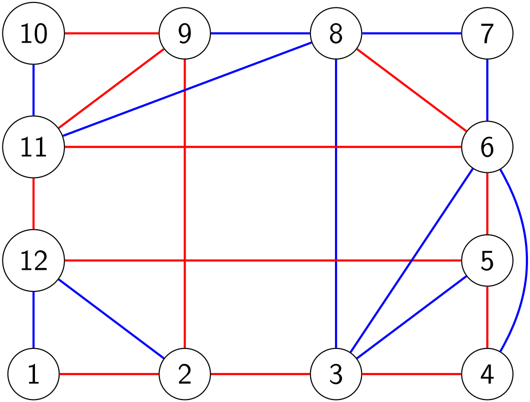

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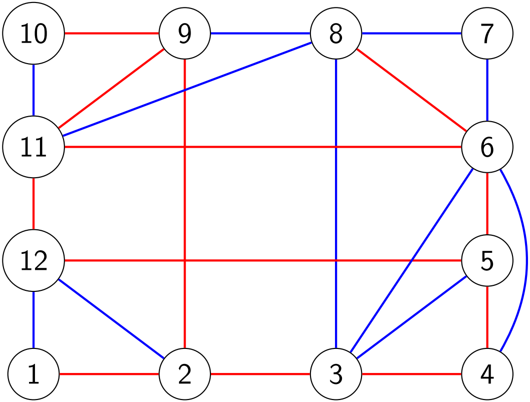

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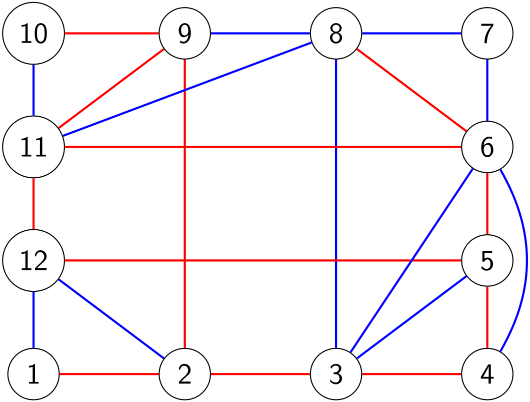

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General Theorem

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This thm generalize easily:

Thm Let $G = (V, E)$. If V can be partitioned into $R(k) - 1$ ind. sets then $\exists\ \mathrm{COL}\colon E \rightarrow [2]$ with no mono K_k.

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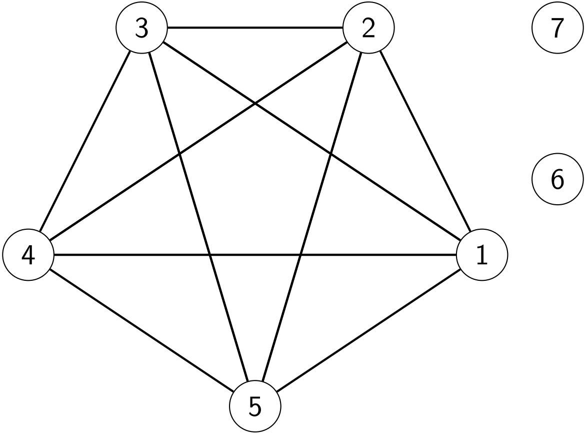

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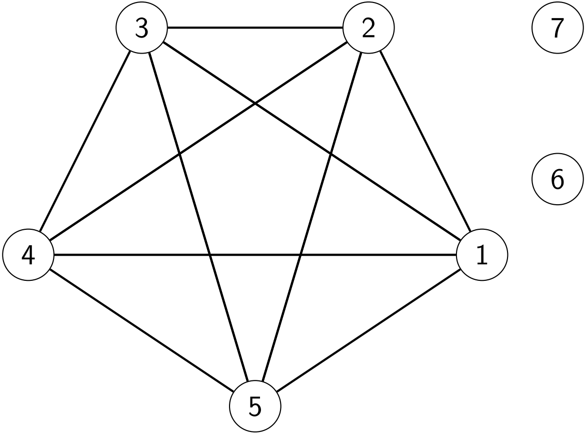

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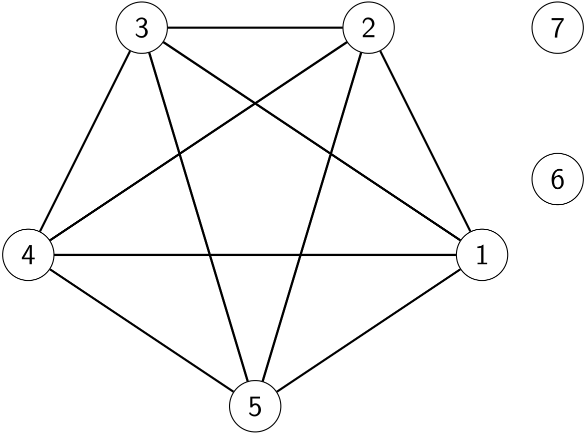

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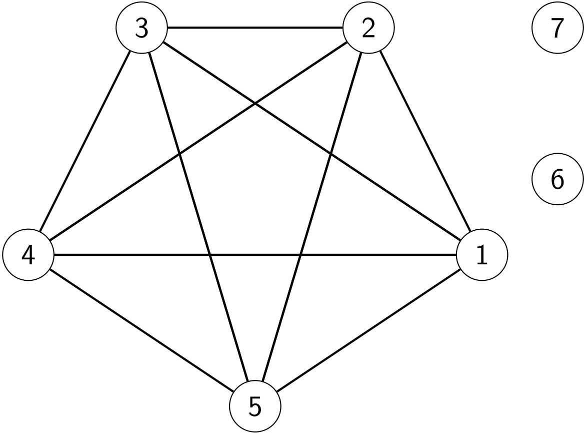

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Those will be our cases.

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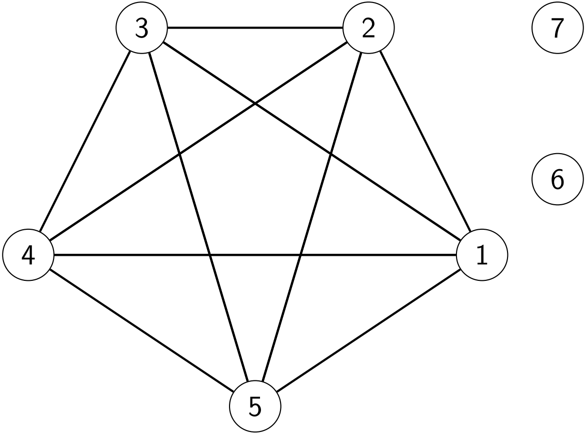

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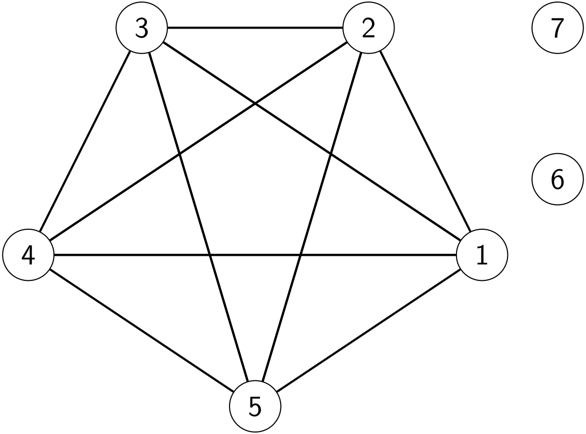

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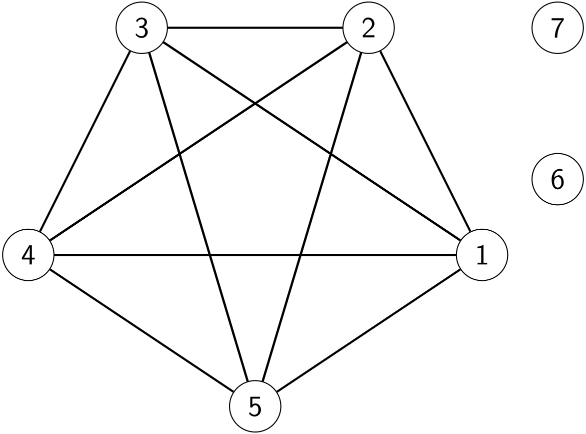

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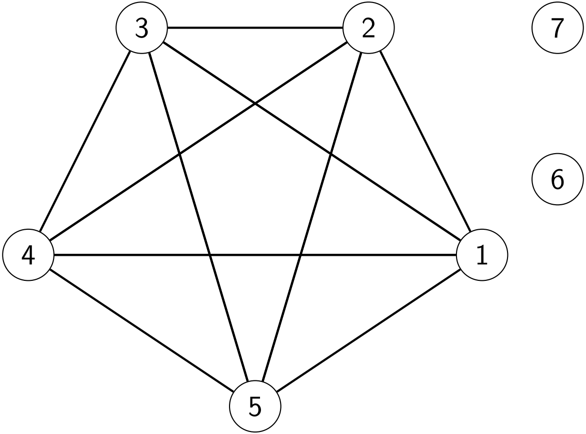

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Use $\{1,6,7\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$.

Case 2b: $\exists i, j, \{i, 6\}, \{j, 7\}$ Both Ind Set

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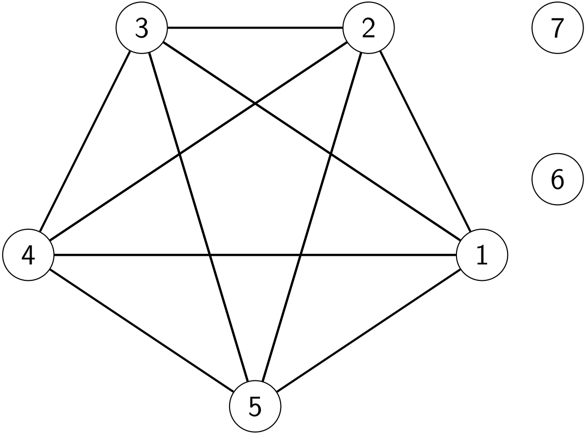

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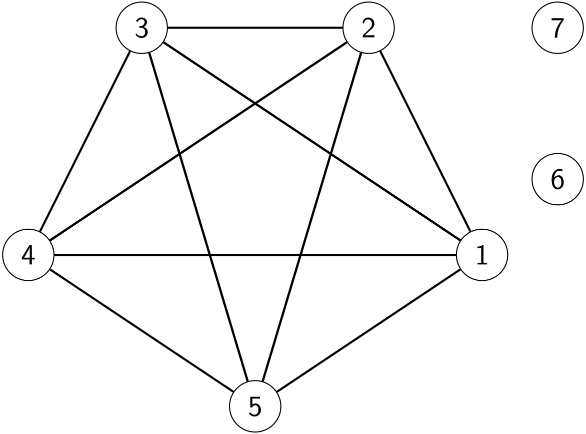

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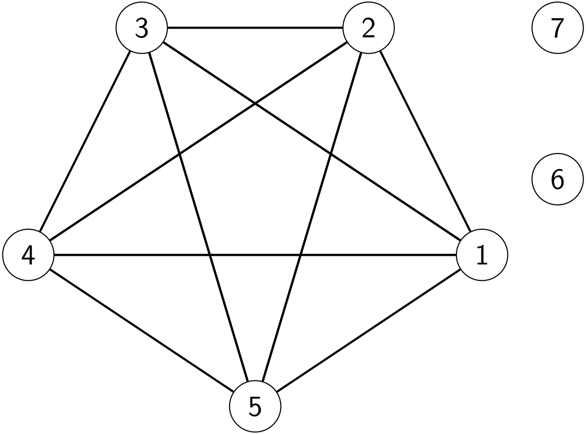

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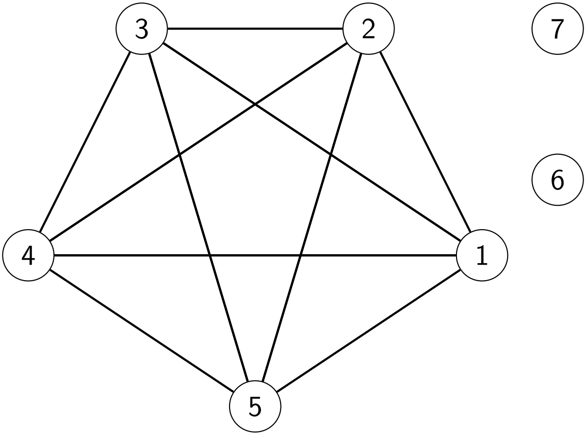

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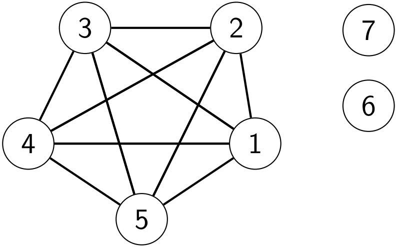

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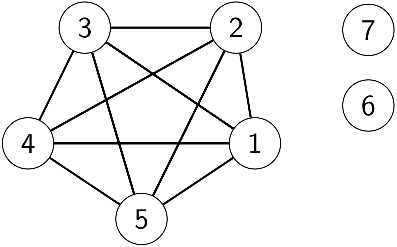

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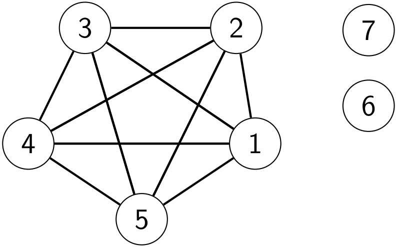

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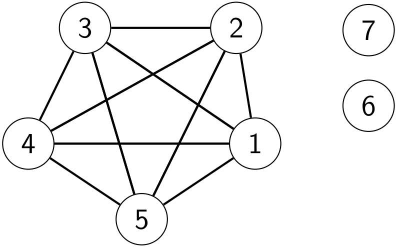

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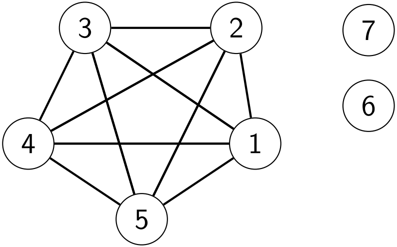

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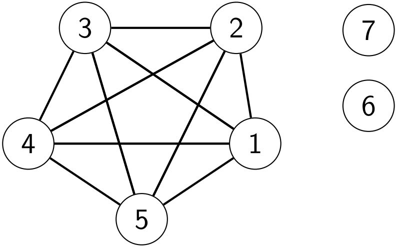

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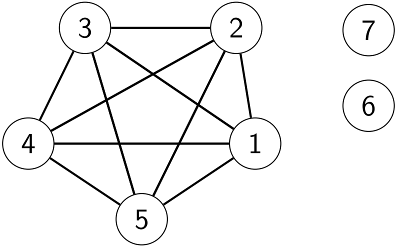

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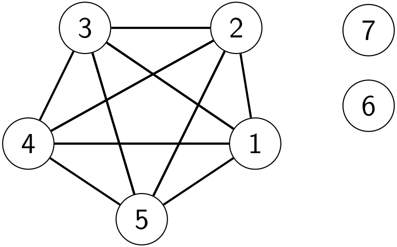

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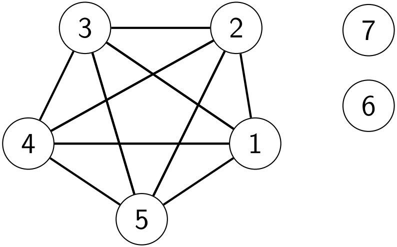

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So $1,2,3,4,5,7$ is a K_6. Contradiction. Case 2c can't happen.

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Note This will require some computer work to go through more cases that we did in our proof.