BILL, RECORD LECTURE!!!!

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From Infinite Ramsey To Finite Ramsey

Exposition by William Gasarch

January 31, 2025

Notation

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3. 2^A is the powerset of A.
4. $\binom{A}{a}$ is the set of all a-sized subsets of A.

Let $\mathrm{COL}\colon \binom{A}{2} \to [2]$. A set $H \subseteq A$ is **homogenous** if COL restricted to $\binom{H}{2}$ is constant. (From now on **homog**.)

Infinite And Finite Ramsey Thm

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We will prove **The Finite Ramsey** from **The Infinite Ramsey.**

Proof of the Finite Ramsey Thm From The Infinite Ramsey Thm

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Say $k = 182$. There is a coloring of $\binom{[10^{100}]}{2}$ with no homog set of size 182.

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Say $k = 182$. There is a coloring of $\binom{[10^{100}]}{2}$ with no homog set of size 182. That seems unlikely.

Lots of Colorings of Finite Graphs

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We use $\mathrm{COL}_0, \mathrm{COL}_1, \ldots$ to form

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We use $\mathrm{COL}_0, \mathrm{COL}_1, \ldots$ to form
$\mathrm{COL}\colon \binom{\mathbb{N}}{2} \rightarrow [2]$.
We will use the inf Ramsey Theory to get a contradiction.

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Let $e_1 = (1, 2)$.

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Answer on the next slide

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COL_0 colors $(1,2)$ **R**

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$\mathrm{COL}(e_1) =$ **R** if $|\{y \colon \mathrm{COL}_y(e_1) = \mathbf{R}\}| = \infty$, **B** OW.

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What about e_2?

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We do the full COL on the next slide.

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Look at COL restricted to $\binom{\{x_1,\ldots,x_k\}}{2}$.

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Look at COL restricted to $\binom{\{x_1,\ldots,x_k\}}{2}$.
By the construction there is an L (actually infinitely many L) such that COL and COL_L agree on $\binom{\{x_1,\ldots,x_k\}}{2}$.

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Hence there is a homog set of size k for COL_L.

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Hence there is a homog set of size k for COL_L.
This is a contradiction since COL_L has no homog sets of size k.

Comments On The Proof

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BILL: So we have proven that, for all k, there is an $n = R(k)$.

STUDENT: Great! what is $R(10)$?

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