$\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$, $\mathsf{R}^n \not\to (\ell_2, \ell_{d'n})$

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1 Introduction

In this paper we present the following theorems:

1. (Szlam [5]) There exists d such that $\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$. (He proved a more general theorem. See his paper for details.)

2. (Conlon & Fox [1]) There exists a constant d' such that $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{d'n}})$. We will just prove the $n = 2$ case in this paper. (They proved a more general theorem. See their paper for details.)

2 There exists d Such That $\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$

Notation 2.1 Let $G_n = (V, E)$ be the graph with $V = \mathsf{R}^n$ and $E = \{(x, y) \colon d(x, y) = 1\}$. Let $c(n)$ be the chromatic number of G_n.

It is well known that $5 \le c(2) \le 7$.
The following are known:

Theorem 2.2

1. *(Larman and Rogers [3])* $c(n) \le (3 + o(1))^n$
2. *(Raigorodskii [4])* $c(n) \le (1.239\ldots + o(1))^n$
3. *(Frankl and Wilson [2])* $c(n) \ge (1 + o(1))(1.2)^n$.

We give the proof by Frankl and Wilson and then use the result to obtain $\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$.

2.1 First Set System Lemma of Frankl and Wilson

Theorem 2.3 *Let $k, n, s \in \mathsf{N}$. Let p be a prime. Let $\mu_0, \ldots, \mu_s$ be distinct element of $\{0, \ldots, p-1\}$. Assume $k \equiv \mu_0 \pmod p$. Let $F_1, \ldots, F_L \in \binom{[n]}{k}$ be such that:*

$$\forall 1 \le i < j \le s)(\exists \mu \in \{\mu_1, \ldots, \mu_s\})|F_i \cap F_j| \equiv \mu \pmod p)].$$

Then $L \le \binom{n}{s}$.

Proof: Let us choose $0 \le a_i < p$ for $0 \le i \le s_0$ [I THINK s_0 SHOULD BE S, BUT PLEASE CHECK] in such a way that for every integer x we have

$$\prod_{i=1}^{s}(x - \mu_i) \equiv \sum_{i=0}^{s} a_i \binom{x}{i} \pmod p.$$

BILL TO BILL- KELIN REC LHS-1 and RHS-0. CHECK THIS. Page 36 of paper.

∎

2.2 Second Set System Lemma of Frankl and Wilson

We will state and prove a Theorem similar to Theorem 2.3 later. Neither theorm implies the other.

Before that we will state and prove a weaker version which is called *the Oddtown Theorem.*

Theorem 2.4 *Let $k, n \in \mathsf{N}$ such that $k \le n$ and k is odd. Let $F_1, \ldots, F_L \in \binom{[n]}{k}$ be such that:*

$$(\forall 1 \le i < j \le s)[|F_i \cap F_j| \equiv 0 \pmod 2].$$

Then $L \le n$.

Proof: For $1 \le i \le L$ let f_i be the bit vector for F_i. Note that f_i is a vector of n bits, k of which are 1's. Let

$$f_i = (f_{i1}, \ldots, f_{in}).$$

Note that $f_{ij} = 1$ iff $j \in F_i$.

We view the f_i's as n-dimensional vectors over $F_2 = \{0, 1\}$ so the arithmetic is mod 2.

We show that the f_i's are linearly independent, hence there are at most n of them, so $L \le n$.

Claim The f_i's are linearly independent $\pmod 2$.

Proof:

$$\begin{aligned} f_i \cdot f_j = & \ f_{i1}f_{j1} + f_{i2}f_{j2} + \cdots f_{in}f_{jn} \\ = & \ |F_i \cap F_j|. \end{aligned}$$

Since $|F_i \cap F_j|$ is even, and $|F_i|$ is odd, we have

$$f_i \cdot f_j \pmod 2 = \begin{cases} 0 \text{ if } i \ne j; \\ 1 \text{ if } i = j. \end{cases} \tag{1}$$

Let $\lambda_1, \ldots, \lambda_L$ be such that

$$\lambda_1 f_1 + \cdots + \lambda_L f_L = 0.$$

Let $1 \le i \le L$. Dot both sides by f_i to get $\lambda_i = 0$

Hence, for every $1 \le i \le L$, $\lambda_i = 0$.

End of Proof of Claim ▮

Note 2.5 Theorem 2.4 still holds if we have $|F_i|$ odd rather than $|F_i| = k$. We state it in the weaker form since then it is a case of Theorem 2.6

Theorem 2.6 *Let $k, n \in \mathsf{N}$. Let q be a prime power.*

1. *Let $F_1, \ldots, F_L \in \binom{[n]}{k}$ be such that:*

$$(\forall 1 \le i < j \le s)[|F_i \cap F_j| \not\equiv k \pmod q].$$

Then $L \le \binom{n}{q-1}$.

2. *If* $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{[n]}{k}$ *then there exists* $1 \le i < j \le \binom{n}{q-1}+1$ *with* $|F_i \cap F_j| \equiv k \pmod{q}$. *(This is the contrapositive of Part 1.)*

3. *If* $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{[n]}{2q-1}$ *then there exists* $1 \le i < j \le \binom{[n]}{q-1}+1$ *with* $|F_i \cap F_j| = q-1$. *(This is Part 2 with* $k = 2q-1$ *coupled with the observation that if* $|F_i \cap F_j| \equiv 2q-1 \pmod{q}$ *then* $|F_i \cap F_j| = q-1$ *since otherwise* $F_i = F_j$*.)*

Proof:
BILL TO BILL - FILL I LATER.
▮

2.3 The Chromatic Number of R^n

Theorem 2.7

1. *For all* n,

$$c(n) > \left\lceil \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1}+1} \right\rceil$$

See the on the next page for value of $c(n)$ *, and see Figure 1 for a graph of* n *vs* $c(n)$*. The curve fitting yields* $c(n) = 2.54 \times 2^{0.266n}$.

2. *There exists* d *such that, for all* n, $c(n) \ge 2^{dn}$.

Proof:
1) Let $S \subseteq \mathsf{R}^n$ be all of the vectors such that

- $n - 2q - 1$ of the components are 0.
- $2q - 1$ of the components are $\frac{1}{\sqrt{2q}}$.

Let $F\colon S \to \binom{[n]}{2q-1}$ by viewing each vector in S as a bit vector though with $\frac{1}{\sqrt{2q}}$ instead of 1.
Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u,v) = 2 - \frac{f+1}{q}$. Hence $d(u,v) = 1$ iff $|F(u) \cap F(v)| = q-1$.
Proof of Claim: Assume $|F(u) \cap F(v)| = f$ then:

- There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.
- There are $2q - 1 - f$ coordinates where u has $\frac{1}{2q}$ and v has 0.
- There are $2q - 1 - f$ coordinates where v has $\frac{1}{2q}$ and u has 0.
- There are $n - 4q + f + 2$ coordinates where u and v are both 0.

Hence $d(u,v) = 2 \times (2q-1-f) \times \frac{1}{2q} = \frac{2q-1-f}{q} = 2 - \frac{f+1}{q}$.

End of Proof of Claim

Restrict COL to S. Since $|S| = \binom{n}{2q-1}$ and there are c colors, some color must occur $\ge \binom{n}{2q-1}/c = \binom{n}{q-1} + 1$ times. Let S' be the subset of S that has that color. Since $S' \subseteq \binom{[n]}{2q-1}$ and $|S'| \ge \binom{n}{q-1} + 1$, by Theorem 2.6.3, there exists two elements of S with intersection of size $q-1$. Let those two elements be $F(u)$ and $F(v)$. Since $|F(u) \cap F(v)| = q-1$, by the Claim, $d(u,v) = 1$.

2) We obtain an approximation to the optimal value of c. ∎

n	$\lceil c(n) \rceil$	q^* (prime power)
2	0	2
3	1	2
4	1	2
5	2	2
6	3	2
7	5	2
8	7	2
9	9	2
10	11	2
11	14	2
12	17	2
13	21	2
14	25	2
15	29	2
16	37	3
17	46	3
18	56	3
19	68	3
20	82	3
21	97	3
22	114	3
23	139	4
24	171	4
25	209	4
26	253	4
27	304	4
28	362	4
29	428	4
30	523	5
31	641	5
32	780	5
33	943	5
34	1131	5
35	1349	5

n	$\lceil c(n) \rceil$	q^* (prime power)
36	1599	5
37	1884	5
38	2209	5
39	2577	5
40	3135	7
41	3919	7
42	4865	7
43	6000	7
44	7355	7
45	8964	7
46	10865	7
47	13102	7
48	15722	7
49	18779	7
50	22535	8
51	27542	8
52	33497	8
53	40549	8
54	48867	8
55	58641	8
56	70083	8
57	84511	9
58	103062	9
59	125147	9
60	151340	9
61	182296	9
62	218756	9
63	261555	9
64	311640	9
65	370073	9
66	438045	9
67	523110	11
68	645540	11
69	793477	11
70	971604	11
71	1185357	11
72	1441022	11
73	1745853	11
74	2108200	11
75	2537648	11
76	3045178	11
77	3643338	11
78	4346438	11

n	$\lceil c(n) \rceil$	q^* (prime power)
79	5170762	11
80	6134803	11
81	7432409	13
82	9127519	13
83	11173342	13
84	13635265	13
85	16589572	13
86	20125055	13
87	24344824	13
88	29368359	13
89	35333807	13
90	42400569	13
91	50752196	13
92	60599636	13
93	72184861	13
94	85784907	13
95	101716390	13
96	120340518	13
97	142068666	13
98	172041626	16
99	212522008	16
100	261802474	16

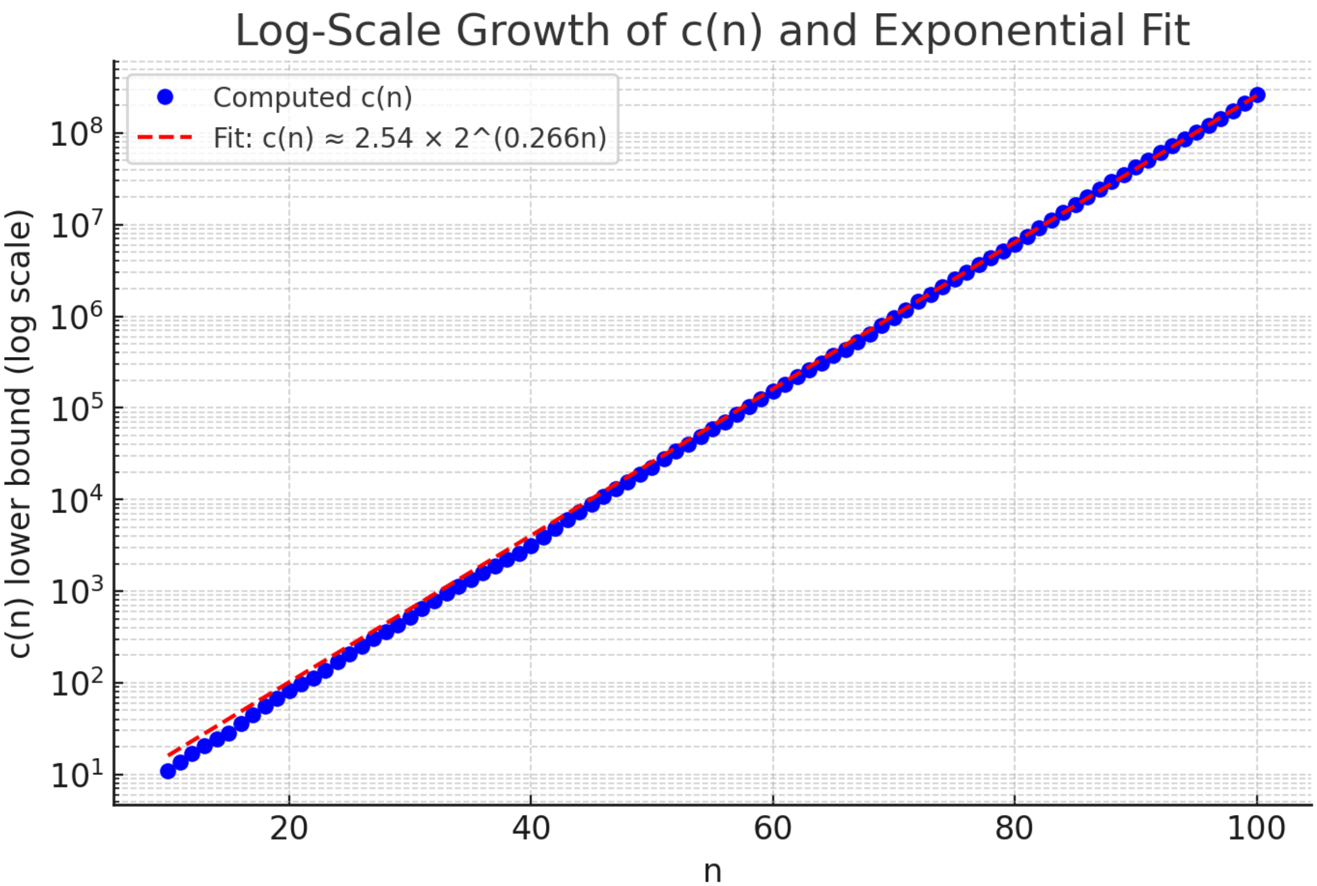

Figure 1: Graph of n vs $c(n)$

2.4 There exists d Such That $\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}}))$

Theorem 2.8 *(Szlam [5]) There exists d such that $\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$.*

Proof: By Theorem 2.7 there exists d such that $c(n) > 2^{dn}$. Thats the d that we use. Let $m = 2^{dn}$.

We will need the following notation: $\vec{1}$ is the vector $(1, 0, \ldots, 0)$ in R^n.

Let $\mathrm{COL}\colon \mathsf{R}^n \rightarrow [2]$.

Case 1 There is a BLUE ℓ_m. Done

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathsf{R}^n \rightarrow [m]$ as follows:

Given point $p \in \mathsf{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED. Color p with the least such i.

By Theorem 2.7 there exists points $u, v \in \mathsf{R}^n$ and $1 \le i \le m$ such that $d(u, v) = 1$ and u, v are the same color. Hence $u + i\vec{1}$ and $v + i\vec{1}$ are both RED. Since $d(u, v) = 1$, $d(u + i\vec{1}, v + i\vec{1}) = 1$. Hence $u + i\vec{1}$ and $v + i\vec{1}$ form a RED ℓ_2. ∎

2.5 While We're Here, Constructive Ramsey Lower Bounds

KELIN- BEFORE ADDING THIS SECTION TO THE MONOGRAPH I WILL GIVE IT MORE CONTEXT.

Frankl and Wilson also used Theorem 2.6 to obtain a constructive lower bound on the Ramsey number $R(k)$.

Theorem 2.9

1. *Let $n \in \mathsf{N}$. Let p be a prime (though we will also use that its a prime power). Let $G = (V, E)$ be defined as follows.*
   - *V is $\{F \subseteq [n] \colon |F| = p^2 - 1\}$. Note that $|V| = \binom{n}{p^2-1}$.*
   - *E is (F, F') such that $|F \cap F'| \not\equiv -1 \pmod{p}$.*

   Then G contains no $\binom{n}{p-1}$-clique or $\binom{n}{p-1}$-ind. set.

2. *$R(k) \ge 2^{(1+o(1)\log^2 k/4 \log\log k}$ with a constructive proof. (Note that this use of k is different than the use of k in Theorem 2.3, 2.6, and the first part of this theorem.)*

Proof:

1a) Let $F_1, \ldots, F_L$ be a complete subgraph of G. By the definition of G,

- $F_1, \ldots, F_L \in \binom{[n]}{p^2-1}$.
- $(\forall 1 \le i < j \le L)[|F_i \cap F_j| \not\equiv -1 \pmod{p}$.

By Theorem 2.6, with $q = p$ and $k = p^2 - 1$, we obtain $L \le \binom{n}{p-1}$.

1b) Let $F_1, \ldots, F_L$ be an independent set in G.
By the definition of G,

- $F_1, \ldots, F_L \in \binom{[n]}{p^2-1}$.
- $(\forall 1 \le i < j \le L)[|F_i \cap F_j| \equiv -1 \equiv p - 1 \pmod{p}$.

By Theorem 2.3, with $k = p^2 - 1$, $\mu_0 = p - 1$, $\mu_1 = p - 1$, we obtain $L \le \binom{n}{2}$.

KELIN: THE ABOVE LINE DOES NOT WORK SINCE $\mu_0 = \mu_1$. THE PAPER DOES NOT USE THEOREM 1. THE PAPER INSTEAD REFERS TO EQUATION (2) WHICH IS FROM A RESULT BY RAY-CHADHUIRI AND WILSON. BUT THEY LATER SAY, RIGHT AFTER THEOREM 1 *Clearly Theorem 1 generalizes (2).* HENCE I THOUGHT I COULD AVOID USING RAY-CHAD... .

TO BILL: theorem 2 implies (2) if you take $p > k$ in theorem 2.

ALSO, $\binom{n}{2}$ SEEMS WAY TO GOOD A BOUND TO GET- FAR MORE THAN WE NEED. I SUSPECT MY TWO CONFUSIONS ARE RELATED AND WHEN YOU FIGURE OUT ONE, YOU WILL FIGURE OUT THE OTHER.

TO BILL: I think you misused (2). (2) does not take mods. We apply (2) on the set $\{|F_i \cap F_j| : 1 \le i < j \le L\}$ which is a subset of $\{0, 1, \ldots, q^2 - 1\}$.

2) Setting $n = p^3$ we obtain the result.

KELIN- WORK THIS OUT.

TO BILL: I couldn't work this out. Also it's not clear to me that exp is base 2 since the exponent cannot be scaled by constant. ∎

3 Lemmas Needed To Show there exists d, $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{dn}})$

We will be 2-coloring the $m \times m$ square and then use that to form a periodic coloring of R^2. Hence we think of coloring the $m \times m$ square with the two horizontal sides identified and the new vertical sides identified. We denote this T_m^2. (The T is for torus.)

BILL- THE PAPER USES $m \times m$. I WILL LATER SAY WHY I USE $m \times m$.

KELIN: WE NEED A PICTURE FOR AN EXAMPLE. YOU CAN DO A COLOR PICTURE OF A colored square. TO BILL: attached

We need several lemmas.

Definition 3.1 Let $t \in \mathsf{R}^+$. Let $P \subseteq T_m^2$.

1. P is t-separated if, for all $p, q \in P$, $d(p, q) \ge t$.
2. P is *maximally t-separated* (1) if P is t-separated and (2) for all $r \notin P$, $P \cup \{r\}$ is not t-separated.

Lemma 3.2 *Let $t \in \mathsf{R}^+$ and $m \in \mathsf{N}$.*

1. *There exists $P \subseteq T_m^2$ that is maximally t-seperated.*

2. *If $P \subseteq T_m^2$ is maximally t-seperated then $|P| \le \frac{(m/t)^2}{\pi}$.*

3. *If $P \subseteq T_m^2$ is maximally $\frac{1}{3}$-seperated then $|P| \le (1.7m)^2$. This follows from Part 2.*

Proof:
1) A greedy algorithm forms a maximally t-seperated set.
2) Let $p \in P$. Then there is no element of P inside the circle centered at p of radius t. This circle has area πt^2. The set T_m^2 has area m^2. Hence

$|P| \times \pi t^2 \le m^2$, so $|P| \le \frac{(m/t)^2}{\pi}$. ▮

Lemma 3.3 *Let $t \in \mathsf{R}^+$. Let $S \subseteq \mathsf{R}^2$ be t-seperated. Let $\vec{p} \in \mathsf{R}^2$. Let $s \ge 0$. The number of points of S within s of $\vec{p}$ is at most $(2s/t+1)^2$.*

Proof: Let T be the set of points within t of $\vec{p}$. For every $\vec{q} \in T$ we look at the circle centered at $\vec{q}$ of radius $t/2$ (we can't use radius t since then the circles would not be disjoint). These circles have no other points of T in them and are disjoint. These circles have area $\pi(t/2)^2$. The union of these circles is contained in the circle around $\vec{p}$ of radius $s+t/2$ which has area $\pi(s+t/2)^2$. Hence

$|T| \times \pi t^2/4 \le \pi(s+t/2)^2$

$|T| \times (t/2)^2 \le (s+t/2)^2$

$|T| \le (\frac{s+t/2}{t/2})^2 = (2s/t+1)^2$. ▮

Definition 3.4 Assume $S \subseteq \mathsf{R}^2$ or $S \subseteq T_2^m$. If $p \in S$ then V_p is the set of points of R^2 or T_2^m that are closer (or tied) to p then to any other point of S. The *Voronoi Diagram of S* is the set of all the V_p's.

BILL- DO EXAMPLES

1. A NORMAL EXAMPLE

2. AN EXAMPLE WHERE THE VORONOI CELL IS A POLYGON WITH LOTS OF SIDES. I THINK IF THE SET OF POINTS IS A p AND m POINTS ON THE CIRCLE OF RADIUS 1 AROUND x THEN V_p would be a m-sided convex polygon.

Note 3.5 There exists $S \subseteq \mathsf{R}^n$ and an $s \in S$ such that V_p is a convex $|S|$-gon. See BILL-WILL NEED FIGURE NUMBER.

Lemma 3.6 *Let $S \subseteq \mathsf{R}^2$ be a maximal t-separated set. We form the Voronoi diagram of S. The Voronoi cells are $\{V_p\}_{p \in S}$.*

1. *If $x \in V_p$ then $d(x,p) \le t$.*

2. *If $p, p' \in V_p$ then $d(p,p') \le 2t$. (This follows from Part 1.)*

3. *If $p, p' \in S$ and V_p, $V_{p;}$ share a boundary then $d(p,p') \le 2t$.*

4. V_p is convex polygon with ≤ 25 sides.

Proof:
1) Assume, by way of contradiction, that there is an $x \in V_p$ and $d(x,p) > t$. Since $x \in V_p$, $d(x,p)$ is the smallest distance from x to a point of S. Hence x is greater than t away from any point in S. Since S is maximal, $x \in S$ which is a contradiction.
3) Draw a line from p to p'. It will hit a point x that is on both the boundary of V_p and the boundary of $V_{p'}$. By Part 1
$d(p,p') = d(p,x) + d(x,p') \le t + t = 2t$.
4) V_p is a convex polygon. Map each side of V_p to the p' such that V_p and $V_{p'}$ share that side. Using Part 2 we get that the number of sides is bounded above by the number of points of $p' \in S$ such that $d(p,p') \le 2t$. By Lemma 3.3 the number of such points is $\le ((2 \times 2t)/t + 1)^2 = 5^2 = 25$. ∎

BILL- I DO NOT THINK I NEED THE LEMMA BELOW FOR THE THEOREM. THEY NEED TO USE A SET OF SIZE $m/5$ THAT HAS POINTS 5 APART. WE WILL JUST NEED THAT ℓ_m DOES NOT HIT TWO ANALOGOUS VORONOI CELLS FROM DIFF TILES. THIS WILL BE ACCOMPLISHED BY MAKING THE TILES $m \times m$ SINCE THE MAX DISTANCE BETWEEN POINTS OF ℓ_m IS $m-1$. THE PAPER DOES MORE COMPLICATED THINGS

Lemma 3.7 *Let K be a 1-seperated set. Let $s \ge 1$. There is a set $K' \subseteq K$ that is s-separated such that $|K'| \ge |K|/(2s+1)^2$.*

4 There exists d', $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{d'n}})$

Theorem 4.1 *There exists d' such that $\mathsf{R}^2 \not\to (\ell_2, \ell_{2^{d'n}})$.*

Proof: Let P be a maximal $\frac{1}{3}$-separated subset of T_2^m. We create the Voronoi diagram of P.

Let $Q \subseteq P$ be formed by, for each $p \in P$, choose it with probability x (we will determine x later).

Let $S \subseteq Q$ be the set of points $s \in Q$ such that, for all $s' \in Q$, $d(s,s') > 5/3$.

Recall that we have a Voronoi diagram formed by the points in P. Let the Voronoi cells that have a point of S in them be denoted $V_1, \ldots, V_{|S|}$.

We will color each V_i, including boundary, RED. We will color every other point in T_2^m BLUE. We will then use this to periodically color R^2. We view this as tiling the plane with $m \times m$ tiles and coloring all the tiles the same.

We will show that if you take a nine tiles arrange 3×3 then there is no RED ℓ_2 or BLUE ℓ_m with a point in the middle tile. This will suffice.

No RED ℓ_2 This part does not use probability.

Let q, q' both be RED.

Case 1: q, q' are in the same Voronoi cell. By Lemma 3.6.2 $d(q,q') \le 1/3$.

Case 2: q, q' are in the same tile but in different Voronoi cells. Let the Voronoi cells have centers p, p'. Then

$$d(p,p') \le d(p,q) + d(q,q') + d(q',p') \le \frac{1}{3} + 1 + \frac{1}{3} = \frac{5}{3}.$$

But by definition of S, $d(p,p') > \frac{5}{3}$.

Case 3: q, q' are in different tiles but in the analogous Voronoi cells. Let the Voronoi cells have centers p, p'. Since $d(p, p') = m$, $d(q, q') \geq m - \frac{1}{3} > 1$.
Case 4: q, q' are in different tiles and non-analogous Voronoi cells. Since the Voronoi diagram was on a Taurus this is identical to Case 2.

No BLUE ℓ_m

Let $L = (q_1, \ldots, q_m)$ be an ℓ_m. We bound the probability that L is BLUE.

Let $\{p_i\}_{i=0}^{m'-1}$ be such that, for $0 \leq i \leq m' - 1$, $q_i \in V_{p_i}$. We need to bound the probability that V_{p_i} is BLUE. Not so fast! We need to show that all of the V_{p_i} are distinct.

Let $q, q' \in \{q_0, \ldots, q_{m'-1}\}$. Let $\{p, p'\}$ be such that $q \in V_p$ and $q' \in V_{p'}$.
Case 1 q, q' are in the same tile and in the same Voronoi cell. This cannot happen since $d(q, q') \geq 1$ and by Lemma 3.6.2 the diameter of these cells is $2/3$.
Case 2 q, q' are in the different tiles but in analogous Voronoi cells. Two points in analogous cells are at least $m - \frac{2}{3}$ apart. Since $d(q, q') \leq m - 1$, q, q' cannot be in different tiles but in analogous Voronoi cells.

The probability that L is BLUE is the prob that V_{p_1}, V_{p_2}, ..., V_{p_m} are all BLUE.

Let $p \in P$. We determine a lower bound on the probability that V_p is RED. Recall that V_p is RED iff $p \in S$.

BILL TO BILL- I NEED TO FINISH THIS. IT REQUIRES THAT LEMMA ABOUT SIGN PATTERNS. ∎

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